NEW NON-QUADRATIC ORTHOTROPIC YIELD FUNCTION FOR TRIAXIAL STRESS STATE①

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ABSTRACT

A new non-quadratic orthotropic yield function is developed in the present paper. It does not have those limitatioins which existing non-quadratic anisotropic yield functions have, such as being usable only for the plane stress problems and in-plane isotropic sheet metals, and that the directions of principal stress or the exponent in yield function can not be arbitrary, etc. Furthermore all of the material constants involved in this yield function can be determined by performing only uniaxial tension test.

This yield function contains three new parameters, of which each one is present for one principal plane of anisotropy.Their values can be, generally, selected to equal 3. Other methods to determine the value of these parmeters are discussed and given in this paper.

From the regression estimate for the yield stress in five directions of several kinds of titanium metal sheet, it is obtained that the suitable value of exponent in yield function for titanium sheets is 6 or 8. This is confirmed from the use for several plastic deformation problems of titanium sheets.

Key words: orthotropy triaxial stress yield function

1 INTRODUCTION

As is well known, Hill's quadratic function[1] overstimated the influence of anisotropic property of material[2]. And so during the 1970s there were several anisotropic yield functions[3-6] proposed one after another by several authors. A common characteristic of all these proposed yield functions is to adopt the non-quadratic functions instead of quadratic functions.

But, just as Hosford has recently pointed out[7], all non-quadratic yield functions have their own limitations.Very recent literature[8-9] which resolved the problem to accommodate planar anisotropy is not perfect because the methods are still limited to the two-dimensional state of stress.

The purpose of the present work is to search for the form of non-quadratic yield function which can accommodate both the orthotropy and the triaxial stresses in order to meet the practical needs of scientific research and production.

2 FORM OF YIELD FUNCTION FOR THREE DIMENSIONAL STATE OF STRESS

The yield function proposed by the present paper is

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$$f = F[(\sigma_y - \sigma_z)^2 + 3(\tau_{zx}^2 + \tau_{xy}^2) + b_{yz}\tau_{yz}^2]^{\frac{m}{2}} + G[(\sigma_z - \sigma_x)^2 + 3(\tau_{xy}^2 + \tau_{yz}^2) + b_{zx}\tau_{zx}^2]^{\frac{m}{2}} + H[(\sigma_x - \sigma_y)^2 + 3(\tau_{yz}^2 + \tau_{zx}^2) + b_{xy}\tau_{xy}^2]^{\frac{m}{2}} + 2L(\tau_{yz}^2)^{\frac{m}{2}} + 2M(\tau_{zx}^2)^{\frac{m}{2}} + 2N(\tau_{xy}^2)^{\frac{m}{2}} - \frac{2}{3}(F+G+H)\sigma_i^m = 0 \quad (1)$$

where x, y, z are the directions of the principal axes of anisotropy, F, G, H, L, M, N are anisotropic parameters, m is the exponent in yield function, b_{xy}, b_{yz}, b_{zx} are three new parameters, and σ_i is the equivalent stress.

3 TWO DIMENSIONAL STATE OF STRESS

The material constants in Eq.(1) can be determined under the simple state of stress. For this reason, now consider the two-dimensional state of stress as follows:

3.1 *Form of the Yield Function for Two-dimensional State of Stress*

Substituting $\sigma_z = \tau_{zx} = \tau_{yz} = 0$ into Eq.(1) and replacing b_{xy} by b, we obtain

$$f = F(\sigma_y^2 + 3\tau_{xy}^2)^{\frac{m}{2}} + G(\sigma_x^2 + 3\tau_{xy}^2)^{\frac{m}{2}} + H[(\sigma_x - \sigma_y)^2 + b\tau_{xy}^2]^{\frac{m}{2}} + 2N(\tau_{xy}^2)^{\frac{m}{2}} - \frac{2}{3}\ (F+G+H)\ \sigma_i^m = 0 \quad (2)$$

3.2 *Corresponding Flow Rules*

On the basis of plastic potential theory and Drucker's postulate[10], the flow rules corresponding with the yield function in the form of Eq.(2) can be derived as follows:

$$d\varepsilon_x = md\lambda\ \{G\sigma_x(\sigma_x^2 + 3\tau_{xy}^2)^{\frac{m}{2}-1} + H(\sigma_x - \sigma_y)[(\sigma_x - \sigma_y)^2 + b\tau_{xy}^2]^{\frac{m}{2}-1}\}$$

$$d\varepsilon_y \doteq md\lambda\{F\sigma_y(\sigma_y^2 + 3\tau_{xy}^2)^{\frac{m}{2}-1} - H(\sigma_x - \sigma_y)[(\sigma_x - \sigma_y)^2 + b\tau_{xy}^2]^{\frac{m}{2}-1}\}$$

$$d\varepsilon_z = -d\varepsilon_x - d\varepsilon_y = -md\lambda[G\sigma_x(\sigma_x^2 + 3\tau_{xy}^2)^{\frac{m}{2}-1} + F\sigma_y(\sigma_y^2 + 3\tau_{xy}^2)^{\frac{m}{2}-1}]$$

$$d\gamma_{xy} = md\lambda\ \{3F(\sigma_y^2 + 3\tau_{xy}^2)^{\frac{m}{2}-1} + 3G(\sigma_x^2 + 3\tau_{xy}^2)^{\frac{m}{2}-1} + bH[(\sigma_x - \sigma_y)^2 + b\tau_{xy}^2]^{\frac{m}{2}-1} + 2N(\tau_{xy}^2)^{\frac{m}{2}-1}\}\ \tau_{xy} \quad (3)$$

and the equivalent strain increment is

$$d\varepsilon_i = 2md\lambda / [3(F+G+H)\sigma_i^{m-1}] \quad (4)$$

where $d\lambda$ is a positive multiplier dependent on the deformed level.The equivalent stress can be obtained from Eq.(2) as follows:

$$\sigma_i = [\frac{3}{2(F+G+H)}]^{\frac{1}{m}}\ \{F(\sigma_y^2 + 3\tau_{xy}^2)^{\frac{m}{2}} + G(\sigma_x^2 + 3\tau_{xy}^2)^{\frac{m}{2}} + H[(\sigma_x - \sigma_y)^2 + b\tau_{xy}^2]^{\frac{m}{2}} + 2N(\tau_{xy}^2)^{\frac{m}{2}}\}^{\frac{1}{m}} \quad (5)$$

3.3 *Determination of Ratios Between Anisotropic Parameters*

Assuming that the uniaxial tensile specimens are prepared along the directions of principal axis of anisotropy, i.e., x, y as well as the 45(°) direction the tests are performed. For the specimens in the 0(°) or 90(°) directions, only $\sigma_x \neq 0$ or $\sigma_y \neq 0$, and so from Eq.(3) it is obtained that

$$\left.\begin{aligned} \gamma_0 &= d\varepsilon_y / d\varepsilon_z = H/G \\ \gamma_{90} &= d\varepsilon_x / d\varepsilon_z = H/F \end{aligned}\right\} \quad (6a)$$

For the specimen in the 45(°) direction, here $\sigma_x = \sigma_y = \tau_{xy} = \sigma_{45}/2$ and the width strain increment of the specimen would be

$$d\varepsilon_{y'} = (d\varepsilon_x + d\varepsilon_y)\ /2 - d\gamma_{xy}/2$$

Thus, introducing the value of γ_{45}, we can obtain

$$\frac{2N}{H}=\frac{F+G}{H}(\gamma_{45}-1)2^{m-1}-b^{m/2} \qquad (6b)$$

It can be seen that Eq.(6b) as well as Eq.(2) and (1) reduces to Hill's old yield function if m is equal to 2.

3.4 *Expressions for Flow Rules in Principal Stress Coordinate System*

Supposing that σ_1 and σ_2 are the two in-plane principal stresses, and that σ_1 has the greater algebraic value, α is the angle between σ_1 and the 0(°) direction. In addition, introducing a ratio x to express the relative level of σ_2 to σ_1, i.e.

$$x=\sigma_2/\sigma_1 \qquad (\sigma_1>\sigma_2) \qquad (7)$$

Introducing the transformation relationships between the stresses and after arrangement, we obtain

$$\sigma_i=D[\gamma_0 B^{\frac{m}{2}}+\gamma_{90}A^{\frac{m}{2}}+(1-x)C]^{\frac{1}{m}}\sigma_1 \qquad (8)$$

where

$$\left.\begin{aligned}
A&=[3-2x+3x^2+2(1-x^2)\cos2\alpha-(1-x)^2\cos4\alpha]/4\\
B&=[3-2x+3x^2-2(1-x^2)\cos2\alpha-(1-x)^2\cos4\alpha]/4\\
C&=\{\gamma_0\gamma_{90}(\cos^2 2\alpha+\frac{b}{4}\sin^2 2\alpha)^{\frac{m}{2}}+[\frac{(\gamma_0+\gamma_{90})(\gamma_{45}-1)}{2}-\gamma_0\gamma_{90}\times(\frac{b}{4})^{\frac{m}{2}}]\sin^m 2\alpha\}(1-x)^{m-1}\\
D&=\{3/[2(\gamma_0+\gamma_{90}+\gamma_0\gamma_{90})]\}^{\frac{1}{m}}
\end{aligned}\right\} \qquad (9)$$

In addition, if the strain components in principal stress coordinate system are expressed by $d\varepsilon_1$, $d\varepsilon_2$ and $d\gamma_{12}$, introducing the transformation relationships between the strains and after arrangement, we obtain

$$\begin{aligned}
d\varepsilon_i&=\{2[\gamma_0 B^{\frac{m}{2}}+\gamma_{90}A^{\frac{m}{2}}+(1-x)C]^{\frac{m-1}{m}}\times d\varepsilon_1\}/D\ \{\gamma_{90}[1+\cos2\alpha+(1-x)\times\sin^2 2\alpha]A^{\frac{m-2}{2}}+\gamma_0[1-\cos2\alpha+(1-x)\sin^2 2\alpha]\times B^{\frac{m-2}{2}}+2C\}\\
&=\{2[\gamma_0 B^{\frac{m}{2}}+\gamma_{90}A^{\frac{m}{2}}+(1-x)C]^{\frac{m-1}{m}}\times d\varepsilon_2\}/D\{\gamma_{90}[x(1-\cos2\alpha)-(1-x)\times\sin^2 2\alpha]A^{\frac{m-2}{2}}+\gamma_0[x(1+\cos2\alpha)-(1-x)\times\sin^2 2\alpha]B^{\frac{m-2}{2}}-2C\}\\
&=-\{2[\gamma_0 B^{\frac{m}{2}}+\gamma_{90}A^{\frac{m}{2}}+(1-x)C]^{\frac{m-1}{m}}\times d\varepsilon_z\}/D\{\gamma_{90}[1+x+(1-x)\cos2\alpha]\times A^{\frac{m-2}{2}}+\gamma_0[1+x-(1-x)\cos2\alpha]\times B^{\frac{m-2}{2}}\}\\
&=\pm\{2[\gamma_0 B^{\frac{m}{2}}+\gamma_{90}A^{\frac{m}{2}}+(1-x)C]^{\frac{m-1}{m}}\times d\gamma_{12}\}/D\{\gamma_{90}[1+x-2(1-x)\cos2\alpha]\times A^{\frac{m-2}{2}}-\gamma_0[1+x+2(1-x)\cos2\alpha]\times B^{\frac{m-2}{2}}-2E\}\sin2\alpha
\end{aligned} \qquad (10)$$

where

$$E=\{[(\gamma_0+\gamma_{90})(\gamma_{45}-1)-2\gamma_0\gamma_{90}\times(\frac{b}{4})^{\frac{m}{2}}]\times(\sin2\alpha)^{m-2}-2\gamma_0\gamma_{90}(1-\frac{b}{4})(\cos^2 2\alpha+\frac{b}{4}\sin^2 2\alpha)^{\frac{m-2}{2}}\}(1-x)^{m-1}\cos2\alpha \qquad (11)$$

3.5 *Determination of m-Value*

From the calculation based on the theory of crystal slipping, Hosford obtained that the best value of m is 6 or 8 for bcc and fcc metals, respectively[4]. Of course, m-value may also be calculated from experimental data. A method of calculating developed by the author is stated below.

3.6 *Selection of b-Value*

Up to now, the condition of planar

isotropy is represented by that $\gamma_0 = \gamma_{45} = \gamma_{90} = \gamma$. If we examine the present yield function, the yield stress at an arbitrary direction can be calculated from Eq.(8) and (9) as follows:

$$\sigma = \left\{ \frac{\gamma_{90}(1+\gamma_0)}{\gamma_0 B^{m/2} + \gamma_{90} A^{m/2} + C} \right\}^{\frac{1}{m}} \sigma_0 \tag{12}$$

$$\left.\begin{aligned} A &= 1 + \sin^2\alpha\cos 2\alpha \\ B &= 1 - \cos^2\alpha\cos 2\alpha \\ C &= \gamma_0\gamma_{90}(\cos^2 2\alpha + \frac{b}{4}\sin^2 2\alpha)^{\frac{m}{2}} + \\ &\quad [\frac{(\gamma_0+\gamma_{90})(\gamma_{45}-1)}{2} - \gamma_0\gamma_{90} \times \\ &\quad (\frac{b}{4})^{\frac{m}{2}}]\sin^m 2\alpha \end{aligned}\right\} \tag{13}$$

Substituting $\gamma_0 = \gamma_{45} = \gamma_{90} = \gamma$ into the above equation, we obtain

$$\begin{aligned} (\frac{\sigma}{\sigma_0})^m &= (1+\gamma)/\{B^{m/2} + A^{m/2} + \gamma \\ &\quad \{(\cos^2 2\alpha + \frac{b}{4}\sin^2 2\alpha)^{m/2} + [1 - \\ &\quad (\frac{b}{4})^{m/2}]\sin^m 2\alpha\} - \sin^m 2\alpha\} \end{aligned} \tag{14}$$

where A, B were given by Eq.(13). It is seen that, unless $m = 2$, the denominator after the equal sign is not always equal to $1+\gamma$ except when $\alpha = 0$, 45 and 90(°). Obviously, in the range of 0~ 90(°), the maximal values of $(\sigma/\sigma_\circ - 1)$ and their orientations are relative to the b-value. The calculated results for some b-values are listed in Table 1.

It is seen that, in the ranges of listed m, γ and b-values, the maximal $(\sigma/\sigma_\circ - 1)$ values are generally not very great except $b = 0$. It is also seen that, if the accuracy for planar isotropy is required, it is not very good to take the b-value as a certain constant to be independent of γ and m. For this reason, the present paper proposes two models to determine the b-value as follows

$$\left.\begin{aligned} b_1 &= [2^{m-1}\frac{(\gamma_0+\gamma_{90})(\gamma_{45}-1)}{\gamma_0\gamma_{90}}]^{\frac{2}{m}} \\ \text{or}\quad b_1 &= 4(1-\frac{1}{\gamma})^{\frac{2}{m}} \end{aligned}\right\} \tag{15}$$

$$b_2 = 4 - \frac{\sqrt{m}}{2\gamma_{cp}} \tag{16}$$

The first model can not be used for the case where $\gamma_{45} < 1$ (or $\gamma < 1$). The maximal values of $(\sigma/\sigma_0 - 1)$ and their orientation calculated using a variable b-value are listed in Table 2.

It is seen that, in the practical ranges of m and γ-value, the described accuracy for planar isotropy is enough if the b-value is selected from Eq.(16) and the maximal $(\sigma/\sigma_0 - 1)$ value

Table 1 Maximal Values of $(\sigma/\sigma_0 - 1)$ Calculated By Eq.(14)

m	6 (bcc)						8 (fcc)					
b	0		3		4		0		3		4	
γ	(σ/σ_0-1)/%	dir./(°)	(σ/σ_0-1)/%	dir./(°)	(σ/σ_0-1)/%	dir./(°)	(σ/σ_0-1)/%	dir./(°)	(σ/σ_0-1)/%	dir./(°)	(σ/σ_0-1)/%	Dir./(°)
0.5	1.29	17.6 72.4	−1.65	25.0 65.0	−2.89	24.7 65.3	0.80	13.4 76.6	−2.15	26.5 63.5	−3.41	26.2 63.8
1.0	4.60	21.1 68.9	−0.23	27.4 62.6	−2.22	as	3.37	18.6 71.4	−0.57	28.0 62.0	−2.65	as
2.0	9.10	22.0 68.0	1.38	23.9 66.1	−1.52		7.50	20.9 69.1	1.34	24.5 65.5	−1.84	
3.0	11.97	22.1 67.9	2.25	24.1 65.9	−1.16	above	10.41	21.6 68.4	2.40	25.1 64.9	−1.41	above

Table 2 Maximal Values of $(\sigma / \sigma_0 - 1)$ Calculated With A Variable b-Value

b	Eq.(15)						Eq.(16)					
m	6			8			6			8		
γ	b_1	$(\sigma/\sigma_0 - 1)/\%$	dir./(°)	b_1	$(\sigma/\sigma_0 - 1)/\%$	dir./(°)	b_2	$(\sigma/\sigma_0 - 1)/\%$	dir./(°)	b_2	$(\sigma/\sigma_0 - 1)/\%$	dir./(°)
0.5	unsuited for $\gamma < 1$						1.55	−0.26	30.5 59.5	1.17	−0.95	29.8 60.2
1.0	0	4.60	21.1 68.9	0	3.37	18.6 71.4	2.78	0.25	19.4 70.6	2.59	0.36	16.9 73.1
2.0	3.17	0.87	23.8 66.2	3.36	0.24	22.0 68.0	3.39	0.26	23.0 67.0	3.29	0.45	23.7 66.3
3.0	3.49	0.55	24.1 65.9	3.61	0.09	22.1 67.9	3.59	0.22	23.8 66.2	3.53	0.42	25.1 64.9

is less than 0.5% except a few cases.

But, it is only a theoretical assumption that $\gamma_0 = \gamma_{45} = \gamma_{90}$ expresses the plannar isotropy. For this reason, if we want to represent the real behavior of sheet metal more accurately, the *b*-value can also be determined by introducing additional experimental data.

The above discussions are in progress for the planar case. Obviously, owing to the fact that the *b*-value may be flexibly selected, the three *b*-values in eq.(1) are fully unnecessary to keep the same value and may be selected from respective conditions. If nothing needs to be considered, for convenience, the *b*-value may be selected to be 3, as the same as the coefficient in front of other shearing stresses in Eq.(1).

4 CALCULATION OF *m*-VALUE AND *b*-VALUE OF TITANIUM SHEETS

The trade-marks and mechanical properties of several kinds of titanium sheet investigated in the present work are listed in Table 3.

As stated above, the yield stress in different directions can be predicted using Eq.(12). Supposing that σ_j is the experimental data of yield stress, n is the number of experimental data points, $\sigma_{cp}[= (\sigma_0 + 2\sigma_{45} + \sigma_{90}) / 4]$ is an average value of the experimental data and $SD (= \sum [(\sigma - \sigma_j)^2 / n]^{1/2} / \sigma_{cp})$ is the relative root mean square deviations between the predicted values, σ, and the experimental data. Obviously, the calculated SD-value depends on the selected m-value and *b*-value. In order to decrease the influence of experimental error of each experiment on the predicted results, we take Eq.(12) as a regression estimate model and carried out calculation using those data listed in Table 3 (here $n = 5$). The calculated results are listed in Table 4.

It is seen that, for the titanium metal sheets, when the *m*-value is selected to be 6~8 and the *b*-value is calculated from Eq.(15) or (16) the calculated results are relatively good. And, it is unsuitable to select b to equal zero, the calculated results in this way are nearly the same as those where $m = 2$.

5 CONCLUSIONS

(1) The non-quadratic orthotropic yield function proposed in the present paper is usable and accurate, and the material constants involved in this function can be determined by performing only uniaxial tension test.

(2) The three new parameters involved in

Table 3 γ-Value and $\sigma_{0.2}$ of Investigated Titanium Sheets

Materials	γ 0/(°)	22.5/(°)	45/(°)	67.5/(°)	90/(°)	$\sigma_{0.2}$/Mpa 0/(°)	22.5/(°)	45/(°)	67.5/(°)	90/(°)
TA2M, 0.8 mm	2.50	2.24	3.36	3.17	3.30	408.9	424.6	430.5	437.4	440.3
TA2M, 1.0 mm	2.05	2.10	3.50	2.70	2.93	347.3	351.2	360.6	383.2	387.8
TA2M, 1.5 mm	1.73		2.47		2.53	371.7	373.6	379.5	395.2	402.1
TA2M, 2.0 mm	2.60		3.25		3.15	397.2	409.9	418.7	420.7	482.6
TC1M, 0.8 mm	2.06	2.13	3.88	2.95	2.87	548.2	538.4	523.7	537.4	544.3
TC1M, 1.0 mm	1.22		2.20		1.78	516.8	531.5	544.3	574.7	582.5
TC1M, 1.5 mm	1.35		2.72		2.10	511.9		533.5		584.5
TC1M, 2.0 mm	0.69	0.78	1.95	1.92	1.75	622.7	612.9	620.8	654.1	667.8
TC3M, 1.0 mm	1.60	1.90	2.88	2.48	2.25	771.1	778.9	811.3	852.5	865.2

Table 4 The Relative Root Mean Deviations S_D% Between the Predicted Results and the Experimental Data for the Yield Stress of Titanium Sheets

m	2	6				8				10			
b		b_1	b_2	3	0	b_1	b_2	3	0	b_1	b_2	3	0
TA2M, 0.8 mm	3.23	2.42	**2.38**	2.46	5.55	2.46	**2.37**	2.47	4.95	2.51	**2.38**	2.45	4.17
TA2M, 1.0 mm	5.33	**3.83**	**3.77**	3.85	5.99	3.93	3.84	3.93	5.56	4.03	3.91	3.97	4.99
TA2M, 1.5 mm	2.10	**1.91**	**1.91**	**2.06**	4.77	2.31	2.31	2.43	4.40	2.47	2.44	2.51	3.75
TA2M, 2.0 mm	2.87	**2.45**	**2.45**	2.68	5.96	**2.46**	**2.46**	2.72	5.34	2.48	2.48	2.68	4.55
TC1M, 0.8 mm	5.83	**0.91**	1.08	1.77	4.88	**0.87**	1.02	1.64	4.04	1.24	1.33	1.90	5.46
TC1M, 1.0 mm	5.22	3.66	**3.58**	**3.57**	4.64	3.82	3.69	3.67	4.31	3.96	3.78	3.77	4.04
TC1M, 1.5 mm	6.99	4.50	4.50	4.50	4.50	4.62	4.62	4.62	4.62	4.74	4.74	4.74	4.74
TC1M, 2.0 mm	8.94	**0.70**	1.08	1.17	3.14	**0.94**	1.09	1.09	2.40	1.25	1.52	1.67	3.93
TC3M, 1.0 mm	5.95	**4.03**	**4.00**	4.08	5.75	4.08	4.04	4.11	5.32	4.15	4.08	4.12	4.85
$\sum$	46.46	**24.41**	**24.75**	26.14	45.18	25.49	25.44	26.68	40.94	26.83	26.66	27.81	40.48

*: 1. b_1 and b_2 is determined from Eq.(15) and (16), respectively;

2. For the TC1M, 1.5mm sheet there are only three experimental data in 0, 45 and 90(°); so S_D is not relative to the b-value;

3. The best results and next results are expressed by the boldfaced types.

this function can have unequal value, which can be determined from additional experimental data or from Eq. (15) or (16), or selected to be 3.

(3) The suitable value of the exponent m, in yield function for titanium sheet is 6~8, and 6 is much better. The suitable b-value can be determined from Eq. (15) or (16), both are about the same. And, it is unsuitable to select the b-value to be zero.

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