STRESSES AND DISPLACEMENTS IN ORTHOTROPIC LONG STRIP LOADED BY SYMMETRIC LOCAL UNIFORM FORCES①

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ABSTRACT

Nowadays composite materials much used in industry are orthotropic. This article introduces the expressions of components of stresses and components of displacements in a rectangular plate or an infinitely long strip which consists of these materials and is loaded by local uniform forces symmetrically at upper and lower edges.

Key words: orthogonal anisotropic material function of stress components of displacement components of stress

Materials such as glued board etc. and crystal of rhombohedral system all are orthotropic, i. e. there are three symmetrical elastic faces which are perpendicular to each other in every point of these materials, the property of which is symmetric to these faces. The direction perpendicular to the elastic symmetrical face is called elastic principal direction. According to anisotropic theory of elasticity, there are nine independent elastic constants in orthotropic body.

Taking the coordinates x, y, and z along the elastic principal directions, general Hook' s law are[1]:

in plane stress state:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E_1}\sigma_x - \frac{\mu_{21}}{E_2}\sigma_y \\ \varepsilon_y &= \frac{1}{E_2}\sigma_y - \frac{\mu_{12}}{E_1}\sigma_x \\ \gamma_{xy} &= \frac{1}{G_{12}}\tau_{xy} \end{aligned}\right\} \tag{1a}$$

in plane strain state:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1-\mu_{13}\mu_{31}}{E_1}\sigma_x - \frac{\mu_{21}+\mu_{23}\mu_{31}}{E_2}\sigma_y \\ \varepsilon_y &= \frac{1-\mu_{32}\mu_{23}}{E_2}\sigma_y - \frac{\mu_{12}+\mu_{13}\mu_{32}}{E_1}\sigma_x \\ \gamma_{xy} &= \frac{1}{G_{12}}\tau_{xy} \end{aligned}\right\} \tag{1b}$$

where E_1, E_2 are modules of elasticity in tension in directions x and y respectively; μ_{21} is Poision' s ratio of contraction in the y direction under stretching in the x direction and so on; G_{12} is the shear modulus of elasticity deciding the change of angular between x, y directions. There are relations between module of tension and the Poison' s ratio as follows:

$$\left.\begin{aligned} E_1\mu_{21} &= E_2\mu_{12} \\ E_2\mu_{32} &= E_3\mu_{23} \\ E_3\mu_{13} &= E_1\mu_{31} \end{aligned}\right\} \tag{2}$$

For simplicity eq. (1a) can be written as:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E_1}\sigma_x - \frac{\mu_2}{E_2}\sigma_y \\ \varepsilon_y &= \frac{1}{E_2}\sigma_y - \frac{\mu_1}{E_1}\sigma_x \\ \gamma_{xy} &= \frac{1}{G}\tau_{xy} \end{aligned}\right\} \tag{3}$$

Solving the problem in stress and neglecting body force the stress components can be expresseed by stress function for the two kinds of problem as follows:

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2\varphi}{\partial y^2} \\ \sigma_y &= \frac{\partial^2\varphi}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2\varphi}{\partial x\partial y} \end{aligned}\right\} \tag{4}$$

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The stress function should satisfy the compatibility equation as follows:

$$\frac{1}{E_2}\frac{\partial^4\varphi}{\partial x^4}+\left(\frac{1}{G}-\frac{2\mu_1}{E_1}\right)\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{1}{E_1}\frac{\partial^4\varphi}{\partial y^4}=0 \quad (5)$$

Ref. 2 has given the results of stress components in isotropic long strip which is loaded by local uniform forces symmetrically at upper and lower edges(Fig. 1). Now let' s discuss the circumstance of orthotropic strip. Let' s expand in$(-l, l)$ the load q as Fourier series:

$$q=q_0+\sum_{k=1}^{\infty}q_k\cos\frac{k\pi y}{l} \quad (6)$$

where

$$q_0=\frac{1}{l}\int_0^a q\,dy=\frac{q}{l}a \quad (7)$$

$$q_k=\frac{2}{l}\int_0^a q\cos\frac{k\pi y}{l}dy$$

$$=\frac{2q}{l}\frac{1}{\frac{k\pi}{l}}\sin\frac{k\pi a}{l} \quad (8)$$

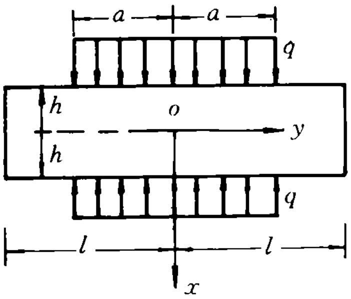

Fig. 1 Rectangular plate loaded by symmetric local uniform forces

By letting $\alpha=k\pi/l$ (9)
and substituting eqs. (7), (8), (9) into eq. (6) we can obtain:

$$q=\frac{q}{l}a+\frac{2q}{l}\sum_{k=1}^{\infty}\frac{1}{\alpha}\sin\alpha a\cdot\cos\alpha y \quad (10)$$

Taking stress function according to q_0 of eq. (6) as:

$$\varphi_0=a_2y^2 \quad (11)$$

and according to $q_k\cos(k\pi y/l)$ as:

$$\varphi_k=f_k\cos(k\pi y/l) \quad (12)$$

and substituting the later into eq. (5), we can obtain an ordinary differential equation for f_k. Its characteristic equation is:

$$\frac{1}{E_2}u^4-\left(\frac{1}{G}-\frac{2\mu_1}{E_1}\right)u^2+\frac{1}{E_1}=0 \quad (13)$$

The solution of eq. (13) is:

$$\left.\begin{matrix}u_1^2\\u_2^2\end{matrix}\right\}=\left[\pm\sqrt{\left(\frac{1}{G}-\frac{2\mu_1}{E_1}\right)^2-4\frac{1}{E_2}\frac{1}{E_1}}+\left(\frac{1}{G}-\frac{2\mu_1}{E_1}\right)\right]/\left(2\cdot\frac{1}{E_2}\right) \quad (14)$$

For orthotropic body consisted by ordinary composite materials:

$$\left(\frac{1}{G}-\frac{2\mu_1}{E_1}\right)^2>4\frac{1}{E_1}\frac{1}{E_2}$$

and

$$\left(\frac{1}{G}-\frac{2\mu_1}{E_1}\right)>\sqrt{\left(\frac{1}{G}-\frac{2\mu_1}{E_1}\right)^2-4\frac{1}{E_1}\frac{1}{E_2}}$$

$\pm u_1$ and $\pm u_2$ are unequal real roots, for which the solution of the ordinary differential equation can be written as:

$$f_k=A_k\mathrm{ch}u_1\alpha x+B_k\mathrm{sh}u_1\alpha x+C_k\mathrm{ch}u_2\alpha x+D_k\mathrm{sh}u_2\alpha x$$

in above expression we have let:

$$\alpha=k\pi/l$$

Substituting f_k into eq. (12) and considering symmetry of this problem we have:

$$\varphi_k=(A_k\mathrm{ch}u_1\alpha x+C_k\mathrm{ch}u_2\alpha x)\cos\alpha y \quad (15)$$

Then the stress function for this problem is:

$$\varphi=\varphi_0+\sum_{k=1}\varphi_k$$

$$=a_2y^2+\sum_{k=1}\cos\alpha y(A_k\mathrm{ch}u_1\alpha x+C_k\mathrm{ch}u_2\alpha x) \quad (16)$$

Substituting eq. (16) into eq. (4) we obtain:

$$\left.\begin{aligned}\sigma_x&=2a_2-\sum_{k=1}\alpha^2\cos\alpha y(A_k\mathrm{ch}u_1\alpha x+C_k\mathrm{ch}u_2\alpha x)\\\sigma_y&=\sum_{k=1}\alpha^2\cos\alpha y(u_1^2A_k\mathrm{ch}u_1\alpha x+u_2^2C_k\mathrm{ch}u_2\alpha x)\\\tau_{xy}&=\sum_{k=1}\alpha^2\sin\alpha y(u_1A_k\mathrm{sh}u_1\alpha x+u_2C_k\mathrm{sh}u_2\alpha x)\end{aligned}\right\} \quad (17)$$

From Fig. 1 the boundary conditions of stresses are:

$x=\pm h$: $\tau_{xy}=0$

$$\sigma_x=-q$$

$$=-\frac{q}{l}\left(a+\sum_{k=1}\frac{1}{\alpha}\right.$$

$\times 2\sin\alpha a\cos\alpha y)$

Substituting τ_{xy} and σ_x of eq. (17) into it, we can obtain:

$$\left.\begin{aligned} q &= -qa/(2l) \\ A_k &= -\frac{q}{l}\frac{2u_2\sin\alpha a\cdot \mathrm{sh}u_2\alpha h}{\alpha^3 A} \\ C_k &= \frac{q}{l}\frac{2u_1\sin\alpha a\cdot \mathrm{sh}u_1\alpha h}{\alpha^3 A}\end{aligned}\right\} \tag{18}$$

in which we have let:

$$A = u_1\mathrm{sh}u_1\alpha h\cdot\mathrm{ch}u_2\alpha h - u_2\mathrm{ch}u_1\alpha h\cdot\mathrm{sh}u_2\alpha h \tag{19}$$

Substituting obtained unknown constants back into eq. (17) we obtain the expressions of stresses in orthotropic strip in Fig. 1:

$$\left.\begin{aligned} \sigma_x &= -\frac{qa}{l}+\frac{2q}{l}\sum_{k=1}^{\infty}\frac{\sin\alpha a}{\alpha A}(u_2\mathrm{sh}u_2\alpha h\cdot\mathrm{ch}u_1\alpha x \\ &\quad - u_1\cdot\mathrm{sh}u_1\alpha h\cdot\mathrm{ch}u_2\alpha x)\cos\alpha y \\ \sigma_y &= \frac{2q}{l}\sum_{k=1}^{\infty}\frac{\sin\alpha a}{\alpha A}(-u_1^2\cdot u_2\cdot\mathrm{sh}u_2\alpha h\cdot\mathrm{ch}u_1\alpha x \\ &\quad + u_2^2\cdot u_1\cdot\mathrm{sh}u_1\alpha h\cdot\mathrm{ch}u_2\alpha x)\cos\alpha y \\ \tau_{xy} &= \frac{2q}{l}\sum_{k=1}^{\infty}\frac{\sin\alpha a}{\alpha A}(-u_1\cdot u_2\cdot\mathrm{sh}u_2\alpha h\cdot\mathrm{sh}u_1\alpha x \\ &\quad + u_2\cdot u_1\cdot\mathrm{sh}u_1\alpha h\cdot\mathrm{sh}u_2\alpha x)\cdot\sin\alpha y\end{aligned}\right\} \tag{20}$$

where A, α, have given in eqs. (19) and (9) respectively; u_1, u_2 are the unequal real roots of eq. (13) and have been expressed in eq. (14).

Substituting obtained expressions of stresses (20) into general Hook' s law (3) we can obtain the components of strains ε_x, ε_y, γ_{xy}, for plane stress state. Substituting obtained ε_x, ε_y into the former two expressions of the following geometric eqs.:

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{aligned}\right\} \tag{21}$$

and integrating them we can obtain the components of displacements of any point in x, y directions respectively in orthotropic strip in Fig. 1:

$$\begin{aligned} u &= -\frac{1}{E_1}\frac{qa}{l}x+\left(\frac{u_2}{E_1}+\frac{\mu_2}{E_2}u_1^2u_2\right)\frac{1}{u_1}\frac{2q}{l} \\ &\quad\times\sum_{k=1}^{\infty}\frac{\sin\alpha a}{\alpha^2 A}\cdot\mathrm{sh}u_2\alpha h\cdot\mathrm{sh}u_1\alpha x\cdot\cos\alpha y \\ &\quad -\left(\frac{u_1}{E_1}+\frac{\mu_2}{E_2}u_2^2u_1\right)\frac{1}{u_2}\frac{2q}{l}\sum_{k=1}^{\infty}\frac{\sin\alpha a}{\alpha^2 A} \\ &\quad\times\mathrm{sh}u_1\alpha h\cdot\mathrm{sh}u_2\alpha x\cdot\cos\alpha y+f_1(y)\end{aligned} \tag{22}$$

$$\begin{aligned} v &= \frac{\mu_1}{E_1}\frac{2q}{l}y-\left(\frac{1}{E_2}u_1^2u_2+\frac{\mu_1}{E_1}u_2\right)\frac{2q}{l} \\ &\quad\times\sum_{k=1}^{\infty}\frac{\sin\alpha a}{\alpha^2 A}\cdot\mathrm{sh}u_2\alpha h\cdot\mathrm{ch}u_1\alpha x\cdot\sin\alpha y \\ &\quad+\left(\frac{1}{E_2}u_2^2u_1+\frac{\mu_1}{E_1}u_1\right)\frac{2q}{l}\sum_{k=1}^{\infty}\frac{\sin\alpha a}{\alpha^2 A} \\ &\quad\times\mathrm{sh}u_1\alpha h\cdot\mathrm{ch}u_2\alpha x\cdot\sin\alpha y+f_2(x)\end{aligned} \tag{23}$$

in which $f_1(y)$, $f_2(x)$ are unknown functions of y and x respectively. To obtain the two functions, substituting γ_{xy} of eq. (20) into the third eq. of eq. (3), then substituting obatined γ_{xy} and u, v of eq. (22), (23) into the third eq. of eq. (21), simplifing them and using eqs. (14) and (2), finally we can obatain:

$$\frac{\mathrm{d}f_1(y)}{\mathrm{d}y}+\frac{\mathrm{d}f_2(x)}{\mathrm{d}x}=0$$

From these we have:

$$f_1(y) = Cy + D$$
$$f_2(x) = -Cx + F$$

Substituting f_1, f_2 into eqs. (22), (23) respectively we can obtain ordinary expressions of components of displacement. By using the conditions of symmetry (Fig. 1):

$x=0,\ y=0:\ u=0,\ v=0$

$y=0:\qquad v=0$

we can obtain $C=D=F=0$. Finally the expressions of components of displacement in orthotropic strip in Fig. 1 are:

$$\begin{aligned} u &= -\frac{1}{E_1}\frac{qa}{l}x+\left(\frac{u_2}{E_1}+\frac{\mu_2}{E_2}u_1^2u_2\right)\frac{1}{u_1}\frac{2q}{l} \\ &\quad\times\sum_{k=1}^{\infty}\left(\frac{\sin\alpha a}{\alpha^2 A}\cdot\mathrm{sh}u_2\alpha h\cdot\mathrm{sh}u_1\alpha x\cdot\cos\alpha y\right) \\ &\quad -\left(\frac{u_1}{E_1}+\frac{\mu_2}{E_2}u_2^2u_1\right)\frac{1}{u_2}\frac{2q}{l}\sum_{k=1}^{\infty}\left(\frac{\sin\alpha a}{\alpha^2 A}\right. \\ &\quad\left.\times\mathrm{sh}u_1\alpha h\cdot\mathrm{sh}u_2\alpha x\cdot\cos\alpha y\right)\end{aligned} \tag{24}$$

$$\begin{aligned} v &= \frac{\mu_1}{E_1}\frac{2q}{l}y-\left(\frac{1}{E_2}u_1^2u_2+\frac{\mu_1}{E_1}u_2\right)\frac{2q}{l} \\ &\quad\times\sum_{k=1}^{\infty}\left(\frac{\sin\alpha a}{\alpha^2 A}\cdot\mathrm{sh}u_2\alpha h\cdot\mathrm{ch}u_1\alpha x\cdot\sin\alpha y\right) \\ &\quad+\left(\frac{1}{E_2}u_2^2u_1+\frac{\mu_1}{E_1}u_1\right)\frac{2q}{l}\sum_{k=1}^{\infty}\left(\frac{\sin\alpha a}{\alpha^2 A}\right. \\ &\quad\left.\times\mathrm{sh}u_1\alpha h\cdot\mathrm{ch}u_2\alpha x\cdot\sin\alpha y\right)\end{aligned} \tag{25}$$

By letting $u_2 = u_1 + \triangle u$ in the first eq. of eq. (20), using L' Hopital rule, differentiating the numerator and the denominator with respect to $\triangle u$ respectively and then letting $\triangle u \to 0$, $u_1 = 1$ we can obtain the component of stress in isotropic strip:

$$\sigma_x = -\frac{qa}{l} + \frac{2q}{l}\sum_{k=1}^{\infty}\frac{1}{\alpha}\frac{\sin\alpha a}{\alpha h + \frac{\text{sh}2\alpha h}{2}}$$
$$\times(-\text{sh}\alpha h\cdot\text{ch}\alpha x - \alpha h\cdot\text{ch}\alpha h\cdot\text{ch}\alpha x - \alpha x\cdot\text{sh}\alpha h\cdot\text{sh}\alpha x)\cos\alpha y$$

Before using the L′ Hôpital rule we have substituted A of eq. (19) into eq. (20). Substituting back $\alpha = k\pi/l$ of eq. (9) into above eq. and letting $x = 0$ we can obtain:

$$\sigma_x = -\frac{qa}{l} - \frac{4q}{\pi}\sum_{k=1}^{\infty}\frac{\sin\frac{k\pi a}{l}}{k} \times \frac{\left(\frac{k\pi h}{l}\text{ch}\frac{k\pi h}{l} + \text{sh}\frac{k\pi h}{l}\right)\cos\frac{k\pi y}{l}}{\text{sh}\frac{2k\pi h}{l} + \frac{2k\pi h}{l}} \quad (26)$$

This is the result given by Ref. 2.

If the strip is infinitely long, i. e. $\pm l \to \pm\infty$, then Fourier series in eqs. (20), (24) and (25) turns into Fourier integral. Having let $\alpha = k\pi/l$ before, now letting $\triangle\alpha = \pi/l$ and $l\to\infty$, the first term in eqs. (20), (24) and (25) vanishes and the later turns into Fourier integral, the components of stress and the components of displacement in infinite long strip are:

$$\left.\begin{aligned}
\sigma_x &= \frac{2q}{\pi}\int_0^{\infty}\frac{\sin\alpha a}{\alpha A}(u_2\text{sh}u_2\alpha h\cdot\text{ch}u_1\alpha x - u_1\text{sh}u_1\alpha h\cdot\text{ch}u_2\alpha x)\cos\alpha y d\alpha\\
\sigma_y &= \frac{2q}{\pi}\int_0^{\infty}\frac{\sin\alpha a}{\alpha A}(-u_1^2u_2\text{sh}u_2\alpha h\cdot\text{ch}u_1\alpha x + u_2^2u_1\text{sh}u_1\alpha h\cdot\text{ch}u_2\alpha x)\cos\alpha y d\alpha\\
\tau_{xy} &= \frac{2q}{\pi}\int_0^{\infty}\frac{\sin\alpha a}{\alpha A}(-u_1u_2\text{sh}u_2\alpha h\cdot\text{sh}u_1\alpha x + u_2u_1\text{sh}u_1\alpha h\cdot\text{sh}u_2\alpha x)\cos\alpha y d\alpha
\end{aligned}\right\} \quad (27)$$

and

$$u = \frac{2q}{\pi}\left(\frac{1}{E_1} + \frac{\mu_2}{E_2}u_1^2\right)\frac{u_2}{u_1}\int_0^{\infty}\frac{\sin\alpha a}{\alpha^2 A}\cdot\text{sh}u_2\alpha h \times\text{sh}u_1\alpha x\cdot\cos\alpha y d\alpha - \frac{2q}{\pi}\left(\frac{1}{E_1} + \frac{\mu_2}{E_2}u_2^2\right)\frac{u_1}{u_2}\int_0^{\infty}\frac{\sin\alpha a}{\alpha^2 A}\cdot\text{sh}u_1\alpha h \times\text{sh}u_2\alpha x\cdot\cos\alpha y d\alpha \quad (28)$$

$$v = -\frac{2q}{\pi}\left(\frac{1}{E_2}u_1^2 + \frac{\mu_1}{E_1}\right)u_2\int_0^{\infty}\frac{\sin\alpha a}{\alpha^2 A}\cdot\text{sh}u_2\alpha h \times\text{ch}u_1\alpha x\cdot\sin\alpha y d\alpha + \frac{2q}{\pi}\left(\frac{1}{E_2}u_2^2 + \frac{\mu_1}{E_1}\right)u_1\int_0^{\infty}\frac{\sin\alpha a}{\alpha^2 A}\cdot\text{sh}u_1\alpha h \times\text{ch}u_2\alpha x\cdot\sin\alpha y d\alpha \quad (29)$$

In all formulae above, A, α, u_1, u_2 are given by eqs. (19), (9) and (14) respectively. The all above formulae are obtained for plane stress state. For plane strain state, substituting the constants of elasticity according to eqs. (1a) and (1b), we can obtain the formulae of the stress and displacement components.

It has to be pointed out that in the previous deduction we have taken directly the operation of differential and integeral of series, but have not shown whether the series is uniform convergence or not. But in fact it is. If we substitute A_k, C_k of eq. (18) into stress function (16), neglect non-series term and then substitute eq. (19) into it, we have:

$$\varphi = \sum_{k=1}^{\infty}\cos\alpha y\Big(-\frac{q}{l} \times\frac{2u_2\sin\alpha a\cdot\text{sh}u_2\alpha h\cdot\text{ch}u_1\alpha x}{\alpha^3(u_1\text{sh}u_1\alpha h\cdot\text{ch}u_2\alpha h - u_2\cdot\text{ch}u_1\alpha h\cdot\text{sh}u_2\alpha h} \times\frac{q}{l}\frac{2u_1\sin\alpha a\cdot\text{sh}u_1\alpha h\cdot\text{ch}u_2\alpha x}{\alpha^3(u_1\text{sh}u_1\alpha h\,\text{ch}u_2\alpha h - u_2\text{ch}u_1\alpha h\text{sh}u_2\alpha h)}$$
$$= \frac{2q}{l}\sum_{k=1}^{\infty}\Big(\frac{\sin\alpha a\cdot\cos\alpha y}{\alpha^3} \times\frac{u_1\text{sh}u_1\alpha h\cdot\text{ch}u_2\alpha x - u_2\text{sh}u_2\alpha h\cdot\text{ch}u_1\alpha x}{u_1\text{sh}u_1\alpha h\cdot\text{ch}u_2\alpha h - u_2\cdot\text{sh}u_2\alpha h\cdot\text{ch}u_1\alpha h}\Big)$$

Because $x \leqslant h$, the fraction of the later part on the right of above expression equals at most to one, the numerator of the former part can always be smaller than 1, and the denominator can increase infinitely with k. Taking numerical series:

$$\sum_{k=1}^{\infty}\frac{1}{k\pi/l}$$

obviously

$$\left|\frac{\cos(k\pi y/l)\cdot\sin(k\pi a/l)}{k\pi/l}\right| \leqslant \frac{1}{k\pi/l}$$

According to Weierstrass′ s criterion the series is uniform convergence. The others can be shown so. The operation for all series above is legal.

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