CHARACTERISTICS OF CRITICAL STATE OF TENSILE INSTABILITY①

Zhou Weixian

Department of Aeronautical Manufacturing Engineering, Northwestern Polytechnical University, Xi′an 710072

ABSTRACT The basic characteristics of strain state at the instant that the deformation of material element bearing every kind of tensile loading enters the instable phase from the stable deformation phase were analysed firstly by the principle of minimum strain energy. It was found that at this instant the strain path of this element is not monotonous. The original strain path ends at this moment and a new strain path begins and becomes effective at the same instant. These analyses indicate, from the physical concept, the scientific basis to distinguish the critical state of instability as well as the analysis method which should be adopted for the kind of problems of tensile instability.

Key words tensile instability critical state deformation characteristic

1 INTRODUCTION

The shortcomings of the exisiting theories[1-5] on the tensile instability lie in the fact that they all do not relate the tensile instability to the formation and development mechanisms of instable deformation. However, these theories are still widely used up till now[6]. The author believes that the tensile instability has its own inherent mechanisms like the compression instability. Why they have not been revealed yet is due to the fact that the existing theories have not grasped the fundamental characteristics of the critical state of the tensile instability and no appropriate analysis methods have been established. This paper aims to explain the former problem theoretically for the convenience to solve the later problem.

2 MEANING OF INSTABILITY

What′s the sign of instability? Is it that the load-bearing ability reaches the limit or that the instable deformation begins? The existing theories have the former in mind, therefore the concept of instability is ambiguous. On the contrary, when the onset of the instable deformation is taken as the sign of instability, the concept of instability is very clear. Because it appears by changing the original deformation pattern of the deformed body, the instable deformation is visible to the naked eye. Moreover, owing to the constraint of the volume constancy condition, the instable deformation occurs in three-dimensions, therefore there exists the sole standard. It will be convenient for the study of the instability to take the onset of the instable deformation as the sign of instability.

3 CHARACTERISTICS OF INSTABLE DEFORMATION

The experimental phenomena show that the instable deformation is one different from the original deformation pattern; it occurs spontaneously when the load-bearing body is deformed to a certain level by changing the original deformation pattern. If the original deformation is uniform, then the non - uniform deformation begins; if the original deformation is non-uniform, then another kind of non-uniform deformation begins.

① Supported by the National Natural Science Foundation of China Received Oct. 20, 1995; accepted Jan. 5, 1996

Whether this understanding is reliable can be proved by the following analyses. Considering the generality of the analyses, take an enough small element as studying object: σ_1 and σ_2 are the principal stresses of plane stress problems; χ is their ratio and $\chi = \sigma_2/\sigma_1$; $P_1 = \sigma_1 f_1$ and $P_2 = \sigma_2 f_2$ are the loads in the corresponding directions; S stands for the progress of the deformation of the material.

3.1 Compression instability

From $P_1 = \sigma_1 f_1$ and $d\varepsilon_1 = -df_1/f_1$, dP_1/dS can be rewritten as

$$\frac{dP_1}{dS} = f_1\frac{d\sigma_1}{dS} + \sigma_1\frac{df_1}{dS}$$
$$= -\left(\frac{d\sigma_1}{d\varepsilon_1} - \sigma_1\right)\frac{df_1}{dS} \tag{1}$$

In eq. (1), $(d\sigma_1/d\varepsilon_1 - \sigma_1)$ is related with the hardening behavior of the material; df_1/dS is related with the change of the geometric dimensions of the deformed body.

In compression deformation, $(d\sigma_1/d\varepsilon_1 - \sigma_1)$ is always larger than zero. Therefore, $dP_1/dS = 0$ only occurs when $df_1/dS = 0$. This means that if the deformation does not terminate at this moment, it must go on in the pattern of bending and there are no other possibilities. Compression deformation turning into bending deformation is inevitably accompanied by the non-uniform changes of stresses and strains along the pole and on the transverse section. According to eq. (1), these non-uniform changes are completely allowed to germinate under the conditions of $dP_1/dS = 0$ and $df_1/dS = 0$. Whether these non-uniform deformations necessarily occur or not has been demonstrated in other works by the principle of mimimum strain energy.

3.2 Tensile instability

In tensile deformation, $df_1/dS \neq 0$, $dP_1/dS = 0$ only occurs in the case of $d\sigma_1/d\varepsilon_1 - \sigma_1 = 0$; and it is not ruled out that df/dS (namely $d\varepsilon_1$) changes non-uniformly at this moment, because this is not contrary to eq. (1). Thus it can be seen that the essential distinction between the tensile instability and the compression instability is: the former is determined by the material behaviors, the latter by the geometric dimensions(ratio of length to diameter).

If the transverse section of the tensiled object is non-uniform, that is, the orginal deformation is non-uniform, then the material which first meets the relation $d\sigma_1/d\varepsilon_1 - \sigma_1 = 0$, namely reaches the condition $dP_1/dS = 0$, is that the deformation is the largest. The deformations of other materials depend on their influence.

If the transverse section is uniform, theoretically the relation $d\sigma_1/d\varepsilon_1 - \sigma_1 = 0$ is satisfied simultaneously anywhere, but how the non-uniform deformation germinates? For this, the strain energy is examined as follows.

Take the plane stress state as an example. The strain energy in unit volume material, dW, can be written as

$$dW = \sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2$$
$$= \sqrt{1-\chi+\chi^2}\,\sigma_1 d\varepsilon_i$$
$$= \sigma_i d\varepsilon_i \tag{2}$$

where the flow rule

$$\frac{d\varepsilon_i}{2\sqrt{1-\chi+\chi^2}} = \frac{d\varepsilon_1}{2-\chi} = \frac{d\varepsilon_2}{2\chi-1}$$
$$= -\frac{d\varepsilon_t}{1+\chi} \tag{3}$$

and the definition of $\sigma_i (= \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} = \sqrt{1-\chi+\chi^2}\,\sigma_1)$ are introduced. If there occurs non-uniform deformation, then $dW/d\varepsilon_i (= \sigma_i)$ is a function of both the deformation level, ε_i, and the length of tensiled object, l. Therefore, the complete differential of $dW/d\varepsilon_i$ is

$$d\left(\frac{dW}{d\varepsilon_i}\right) = \frac{\partial}{\partial l}\left(\frac{dW}{d\varepsilon_i}\right)dl + \frac{\partial}{\partial \varepsilon_i}\left(\frac{dW}{d\varepsilon_i}\right)d\varepsilon_i$$
$$= \left[\left(\frac{\chi-1/2}{1-\chi+\chi^2}\frac{\partial\chi}{\partial l} + \frac{1}{\sigma_1}\frac{\partial\sigma_1}{\partial l}\right)dl + \frac{n}{\varepsilon_i}d\varepsilon_i\right]\sigma_i$$

where the relation $\frac{dW}{d\varepsilon_i} = \sigma_i$ is used, and the hardening equation $\sigma_i = k\varepsilon_i^n$ is introduced. Thus, the partial differential of $\frac{dW}{d\varepsilon_i}$ to l is

$$\frac{\partial}{\partial l}\left(\frac{dW}{d\varepsilon_i}\right) = \left(\frac{\chi-1/2}{1-\chi+\chi^2}\frac{\partial\chi}{\partial l} + \frac{1}{\sigma_1}\frac{\partial\sigma_1}{\partial l} + \frac{n}{\varepsilon_i}\frac{\partial\varepsilon_i}{\partial l}\right)\sigma_i \tag{4}$$

As mentioned above, there occur non-uniform changes in strain increment and stress increment, therefore in eq. (4) $\partial\sigma_1/\partial l = 0$ and $\partial\varepsilon_i/\partial l = 0$. From eq. (3), there is $d\varepsilon_2/d\varepsilon_1 = (2\chi-1)/(2-\chi)$; by using this relation, $\frac{\partial\chi}{\partial l}$ in eq. (4) can be expressed as

$$\frac{\partial\chi}{\partial l} = \frac{(2-\chi)^2}{3d\varepsilon_1}\frac{\partial}{\partial l}d\varepsilon_2 + \frac{(2-\chi)(1-2\chi)}{3d\varepsilon_1}\frac{\partial}{\partial l}d\varepsilon_i \qquad (5)$$

It is clear that if $\frac{\partial}{\partial l}d\varepsilon_1 \neq 0$, then $\frac{\partial\chi}{\partial l}$, $\frac{\partial}{\partial l}d\varepsilon_2$ should not be zero at the same time. For the extreme case of $\frac{\partial}{\partial l}d\varepsilon_2 = 0$, this equation gives the sign of $\frac{\partial\chi}{\partial l}$ directively. $\frac{\partial}{\partial l}d\varepsilon_2 \neq 0$ means that there occurs change in the shape of element, that is, the curvatures of principal stress lines. The states of $\frac{\partial\chi}{\partial l}$ can be determined qualitatively from the equilibrium equation. These cases give that, in the stress field of $\chi > 1/2$, $\frac{\partial\chi}{\partial l}$ and $\frac{\partial}{\partial l}d\varepsilon_1$ are of different signs; in the stress field of $\chi < 1/2$, they have identical signs. Furthermore, considering the positive sign or negative sign before $\frac{\partial\chi}{\partial l}$, at the critical state eq. (4) can be rewritten as

$$\frac{\partial}{\partial l}\left(\frac{dW}{d\varepsilon_i}\right)\Big/\frac{\partial}{\partial l}d\varepsilon_1 = \sigma_i\frac{\chi-1/2}{1-\chi+\chi^2}\frac{\partial\chi}{\partial l}\Big/\frac{\partial}{\partial l}d\varepsilon_1 \leqslant 0$$

The case being equal zero only occurs in the boundary-element given by the external force. All other cases being smaller than zero means that the energy is smaller where $d\varepsilon_i$ is larger, and the non-uniform deformation will germinate and develop.

3.3 Superplastic materials

Following the analyses above, introduing the characteristic equation of superplasticity $\sigma_i = K\dot{\varepsilon}_i^m$ gives

$$\frac{\partial}{\partial l}\left(\frac{d\dot{W}}{d\dot{\varepsilon}_i}\right)\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} = \sigma_i\left(\frac{\chi-1/2}{1-\chi+\chi^2}\frac{\partial\chi}{\partial l}\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} + \frac{m}{\dot{\varepsilon}_i}\frac{\partial\dot{\varepsilon}_i}{\partial l}\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l}\right) \qquad (6a)$$

Because $\frac{\partial\dot{\varepsilon}_i}{\partial l}\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} > 0$, depending on the m-value and the stress state, $\frac{\partial}{\partial l}\left(\frac{d\dot{W}}{d\dot{\varepsilon}_i}\right)\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l}$ may be larger or smaller than zero, that is, at the condition $dP_1/dS = 0$, instable deformation does not necessarily occur. What is the critical m-value when there does not occur instable deformation? This m-value can be estimated as follows: Using the $\dot{\varepsilon}_i \sim \dot{\varepsilon}_2$ relation in superplastic flow rule like that expressed by eq. (3), there is $\frac{1}{\dot{\varepsilon}_i}\frac{\partial\dot{\varepsilon}_i}{\partial l} = \frac{1}{\dot{\varepsilon}_2}\frac{\partial\dot{\varepsilon}_2}{\partial l} + \left(\frac{\chi-1/2}{1-\chi+\chi^2} - \frac{1}{\chi-1/2}\right)\frac{\partial\chi}{\partial l}$, and introducing it into eq. (6a) gives

$$\frac{\partial}{\partial l}\left(\frac{d\dot{W}}{d\dot{\varepsilon}_i}\right)\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} = \sigma_i\left\{\left[\frac{1+m}{1-\chi+\chi^2} - \frac{m}{(\chi-1/2)^2}\right](\chi-1/2)\frac{\partial\chi}{\partial l}\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} + \frac{m}{\dot{\varepsilon}_2}\frac{\partial\dot{\varepsilon}_2}{\partial l}\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l}\right\} \qquad (6b)$$

The no instability condition is $\frac{\partial}{\partial l}\left(\frac{d\dot{W}}{d\dot{\varepsilon}_i}\right)\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} \geqslant 0$. As stated above, in the stress field of $\dot{\varepsilon}_2 \leqslant 0$, namely $\chi \leqslant 1/2$, if there germinates instable deformation, then $\frac{\partial\dot{\varepsilon}_2}{\partial l}\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} \leqslant 0$, $\frac{\partial\chi}{\partial l}\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} \geqslant 0$; in the stress field of $\dot{\varepsilon}_2 \geqslant 0$, namely $\chi \geqslant 1/2$, then $\frac{\partial\dot{\varepsilon}_2}{\partial l}\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} \geqslant 0$, $\frac{\partial\chi}{\partial l}\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} \leqslant 0$. Because $\frac{1}{\dot{\varepsilon}_2}\frac{\partial\dot{\varepsilon}_2}{\partial l}\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l} \geqslant 0$ always, the minimum of $\frac{\partial}{\partial l}\left(\frac{d\dot{W}}{d\dot{\varepsilon}_i}\right)\Big/\frac{\partial\dot{\varepsilon}_1}{\partial l}$ is reached when $\frac{\partial\dot{\varepsilon}_2}{\partial l} = 0$; if the value of

$\frac{\partial}{\partial l}(\frac{dW}{d\dot{\varepsilon}_i})/\frac{\partial \dot{\varepsilon}_1}{\partial l}$ is to be non-negative, then $[\frac{1+m}{1-\chi+\chi^2}-\frac{m}{(\chi-1/2)^2}]$ and $(\chi-1/2)\frac{\partial \chi}{\partial l}/\frac{\partial \dot{\varepsilon}_1}{\partial l}$ must have the same signs; because the latter is non-positive, then the former should also be non-positive. Therefore,

$$m_{\min} \geqslant 4/3(\chi-1/2)^2 \geqslant 1/3 \quad (x = 0)$$

It is not surprising that it is deduced in experiments that only the materials of $m > 0.3$ have superplasticity.

3.4 Hydraulic bulging

Under the action of hydraulic pressure, there is an equilibrium equation

$$p = (\frac{\sigma_1}{R_1} + \frac{\sigma_2}{R_2})t = (1+\rho\chi)\frac{\sigma_1 t}{R_1}, \quad (\chi = \frac{\sigma_2}{\sigma_1}, \ \rho = \frac{R_1}{R_2}) \tag{7}$$

Taking the logarithm of eq. (7) and differentiating to deformation progress S and rearranging gives

$$\frac{dp}{dS} = [\sigma_1\frac{d(\rho\chi)}{d\varepsilon_t} + (1+\rho\chi)(\frac{d\sigma_1}{d\varepsilon_t} + \sigma_1 - \frac{\sigma_1}{R_1}\frac{dR_1}{d\varepsilon_t})]\frac{t}{R_1}\frac{d\varepsilon_t}{dS} \tag{8}$$

In the stable deformation phase, $d\varepsilon_t/dS \neq 0$; therefore, the condition of $dp/dS = 0$ occurs when

$$\sigma_1\frac{d(\rho\chi)}{d\varepsilon_t} + (1+\rho\chi) \times (\frac{d\sigma_1}{d\varepsilon_t} + \sigma_1 - \frac{\sigma_1}{R}\frac{dR_1}{d\varepsilon_t}) = 0 \tag{9}$$

and it is not ruled that abnormal change takes place in $d\varepsilon_t/dS$. If the original stress field and strain field are non-uniform, then the material which first meets eq. (9), namely holds back the continuous increase of the hydraulic pressure is that the deformation is the largest; at the integrated state of $dp/dS = 0$, from eq. (8) other materials can only have the solution of $d\varepsilon_t/dS = 0$, but the abnormal change of the value of the term in the brackets is not ruled out. The non-uniform deformation starting from this moment is different from the original.

In the original uniform stress field, theoretically all parts of the deformed object should satisfy eq. (9) simultaneously. But then how can the non-uniform deformation germinate?

First of all, it is appropriate to make clear what kinds of deformed objects can have uniform stress field and strain field. Obviously, only those deformed objects of uniform geometric shapes can do so, which are nothing more than spherical shells and circular cylinders. Their circular strain increment is $d\varepsilon_1 = dR_1/R_1$; therefore the non-uniform change of $d\varepsilon_t$, i. e $d\varepsilon_1$ along the circumference l can be expressed by

$$\frac{\partial}{\partial l}d\varepsilon_1 = \frac{\partial}{\partial l}(\frac{dR_1}{R_1}) = \frac{1}{R_1}\frac{\partial}{\partial l}dR_1 - \frac{dR_1}{R_1^2}\frac{\partial R_1}{\partial l} \approx -\frac{d\varepsilon_1}{R_1}\frac{\partial R_1}{\partial l} \tag{10}$$

where the higher-order differential is neglected. It can be seen that the curvature radius reduces at local thinning places, which agrees with the physical concepts.

In order to make the strain energy analyses, introducing eq. (7) into eq. (2) gives

$$\frac{dW}{d\varepsilon_i} = \sigma_i = \sqrt{1-\chi+\chi^2}\,\sigma_1 = \frac{\sqrt{1-\chi+\chi^2}}{1+\rho\chi}\frac{pR_1}{t}$$

if there occurs non-uniform deformation, it will also be non-uniform. Therefore, there is

$$d(\frac{dW}{d\varepsilon_i}) = \frac{\partial}{\partial l}(\frac{dW}{d\varepsilon_i})dl + \frac{\partial}{\partial \varepsilon_i}(\frac{dW}{d\varepsilon_i})d\varepsilon_i = \{[\frac{\chi-1/2}{1-\chi+\chi^2}\frac{\partial \chi}{\partial l} - \frac{1}{1+\rho\chi} \times \frac{\partial}{\partial l}(\rho\chi) + \frac{1}{R_1}\frac{\partial R_1}{\partial l} - \frac{1}{t}\frac{\partial t}{\partial l}]dl + \frac{n}{\varepsilon_i}d\varepsilon_i\}\sigma_i$$

where the relations $\partial p / \partial l = 0$ and $\sigma_i = K\varepsilon_i^n$ are introduced. Thus,

$$\frac{\partial}{\partial l}(\frac{dW}{d\varepsilon_i}) = [\frac{\chi-1/2}{1-\chi+\chi^2}\frac{\partial \chi}{\partial l} - \frac{1}{1+\rho\chi}\frac{\partial}{\partial l}(\rho\chi) +$$

$$\frac{1}{R_1}\frac{\partial R_1}{\partial l} - \frac{1}{t}\frac{\partial t}{\partial l} + \frac{n}{\varepsilon_i}\frac{\partial \varepsilon_i}{\partial l}]\sigma_i \tag{11}$$

For a spherical shell, $\rho = 1$, $\chi = 1$; for a circular cylinder, $\rho = 0$, at the moment when the non-uniform deformation germinates, $\frac{\partial \rho}{\partial l} = \rho(\frac{1}{R_1}\frac{\partial R_1}{\partial l} - \frac{1}{R_2}\frac{\partial R_2}{\partial l}) = 0$. Therefore, they all satisfy $\frac{\partial}{\partial l}(\rho\chi) = 0$. $\frac{\partial \chi}{\partial l}$ and $\frac{\partial R_1}{\partial l}$ expressed as eq. (5) and eq. (10), respectively, are related with $\frac{\partial}{\partial l}\mathrm{d}\varepsilon_1$; and $\frac{\chi - 1/2}{1 - \chi + \chi^2}\frac{\partial \chi}{\partial l}/\frac{\partial}{\partial l}\mathrm{d}\varepsilon_1 \leqslant 0$, $\frac{1}{R_1}\frac{\partial R_1}{\partial l}/\frac{\partial}{\mathrm{d}l}\mathrm{d}\varepsilon_1 < 0$. At the moment when the non-uniform deformation starts, ε_i and t are still uniform, therefore

$$\frac{\partial}{\partial l}(\frac{\mathrm{d}W}{\mathrm{d}\varepsilon_i})/\frac{\partial}{\partial l}\mathrm{d}\varepsilon_1 = \begin{cases} \frac{\sigma_i}{R_1}\frac{\partial R_1}{\partial l}/\frac{\partial}{\partial l}\mathrm{d}\varepsilon_1 < 0 \text{(spherical shell)} \\ \sigma_i(\frac{\chi - 1/2}{1 - \chi + \chi^2}\frac{\partial \chi}{\partial l}/\frac{\partial}{\partial l}\mathrm{d}\varepsilon_1 + \frac{1}{R_1}\frac{\partial R_1}{\partial l}/\frac{\partial}{\partial l}\mathrm{d}\varepsilon_1) < 0 \text{(circular cylinder)} \end{cases}$$

As a result, the non-uniform deformation with the form of local curvature radius reducing, namely local swelling can germinate at the condition $\mathrm{d}p/\mathrm{d}S = 0$.

4 CONCLUSIONS

(1) $\mathrm{d}P = 0$ is not the intrinsic sign of instability, but it is the indispensable condition for the germination of abnormal deformation.

(2) Instable deformation is one that occurs spontaneously to follow the strain path leading to minimum strain energy when the deformation of material develops to a certain level; it is always non-uniform.

(3) The basic characteristic of the critical state of instability is that the strain paths at this instant are not monotonous: the original strain path ends at this instant and a new one begins and develops at the same moment.

(4) The methods which can reveal the characteristic above should be adopted to discuss the instability mechanisms, otherwise it is difficult to achieve the goal.

REFERENCES

1 Swift H W. J Mech Phys Solids, 1952, 1(1): 1-18.

2 Томленов А Д. Мех∂ник∂ Процесов Обр∂ботки Мет∂ллов Д∂влением, Москв∂, Н∂у-Тех. Изд, 1963: 158-163.

3 Negroni F, Thomsen E G. Trans ASME Ser B, 1969, 91(3): 659-663.

4 Hill R. J Mech Phys Solids, 1952, 1(1): 19-30.

5 Marciniak Z, Kuczyski K. Int J Mech Sci, 1967, 9(9): 609-620.

6 Zhou Y, Neale K W. Int J Mech Sci, 1995, 37(1): 1-20.

(Edited by Peng Chaoqun)