DYNAMIC MODELLING OF MECHANICAL STRUCTURE *VIA* UNDETERMINED PARAMETER METHOD[①]

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ABSTRACT A technique was presented for dynamic modelling of a mechanical structure. It is required to establish an equivalent lumped parameter dynamical model for a given structure, then by means of undetermined parameter method, the dynamic parameters of the mechanical structure are calculated from measured response data. In order to raise the accuracy of identified parameters, an optimization procedure was used. This technique has been applied to modelling of a drill Z512- 2, good result was obtained.

Key words dynamic modelling mechanical structure undetermined parameter

1 INTRODUCTION

In order to improve a mechanical structure or predict the effect of the change of structural parameters on dynamic characteristics of the mechanical structure, it is necessary to establish a dynamical model of the mechanical structure. A technique for dynamic modelling of a mechanical structure is proposed. The procedure of modelling divides into following main steps: (1) An equivalent linear lumped parameter dynamical model is given by testing and analyzing the structure. (2) By applying the Lagrangian equation of motion, the dynamic equations of the model are formulated. (3) By means of undetermined parameter method, the mass, damping and stiffness matrices are calculated from measured response data. (4) Comparing the above matrices with those obtained from dynamical model, the equivalent dynamic parameters are obtained. (5) The parameters of dynamical model are optimized to improve the accuracy of modelling.

2 UNDETERMINED PARAMETER METHOD

Assuming that the equivalent linear lumped parameter dynamical model of N-degree-freedom for a mechanical structure is given by testing and analyzing in a frequency range of interest, the eqn. of the system can be written as:

$$[M]\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\}=\{f(t)\} \quad (1)$$

where $[M]$—mass matrix, $[C]$—damping matrix, $[K]$—stiffness matrix, $\{X\}$—displacement vector, $\{f(t)\}$—forcing function vector.

Consider the steady-state case and let

$$\{X\}=\{X\}e^{iwt} \text{ and } \{f(t)\}=\{F\}e^{iwt} \quad (2)$$

then Eqn. (1) becomes

$$[([K]-w^2[M])+iw[C]]\{X\}=\{F\} \quad (3)$$

where $\{X\}$ is generally complex. If $\{F\}$ is also complex, then

$$\{X\}=\{XR\}+i\{XI\} \quad (4)$$

$$\{F\}=\{FR\}+i\{FI\} \quad (5)$$

where $\{XR\}$—real component of $\{X\}$, $\{XI\}$—imaginary component of $\{X\}$, $\{FR\}$—real component of $\{F\}$, $\{FI\}$—imaginary component of $\{F\}$.

Eqn. (3) is re-written as

$$\left.\begin{aligned}&[([K]-w^2[M])]\{XR\}-w[C]\{XI\}=\{FR\}\\&[([K]-w^2[M])]\{XI\}+wi[C]\{XR\}=\{FI\}\end{aligned}\right| \quad (6)$$

If the matrices $[M]$, $[C]$ and $[K]$ are all full matrices, it is evident that there are $3N$ undermined parameters to be determined in these matrices. But then, if the matrices $[M]$, $[C]$ and

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[K] are symetric matrices undermined parameters to be solved are $q(=3N(n+1))$. To determine these undermined parameters, the number of equations (p) can not be less than q. As can be seen from Eqn. (6), $2N$ equations can be obtained for each sampling frequency $w_i(i=1, 2, \cdots, l)$. Provided w_i are properly choosen, and $p(=2Nl)$ is larger than or equal to q, linear simultaneous equations for solving undermined parameters of [M], [C] and [K] can be yielded:

$$[A]\{y\}=\{\overline{F}\} \tag{7}$$

where

$$\{y\}=[m_{11}, m_{12}, \cdots, m_{1n}, m_{22}, m_{23}, \cdots m_{2n}, \cdots, m_{nn}, c_{11}, c_{12}, \cdots, c_{1n}, c_{22}, c_{23}, \cdots, c_{2n}, \cdots\cdots, c_{nn}, k_{11}, k_{12}, \cdots, k_{1n}, k_{22}, k_{23}, \cdots, k_{2n}, \cdots\cdots, k_{nn}] \tag{8}$$

where m_{ij} — ij elements of matrix [M], c_{ij} — ij elements of matrix [C], k_{ij} — ij elements of matrix [K], $(i=1, 2, \cdots, n; j=i, i+1, \cdots, n)$

$$\{\overline{F}\}=[\{F\}_1^T, \{F\}_2^T, \cdots, \{F\}^T]^T_{2nl\times 1} \tag{9}$$

and

$$\{F\}_i=[\{FR\}_i^T, \{FI\}_i^T]^T_{2n\times 1} \tag{10}$$

$$[A]^T=\begin{bmatrix} A_{11}^{mT} & A_{12}^{mT} & A_{21}^{mT} & A_{22}^{mT} & \cdots & A_{l1}^{mT} & A_{l2}^{mT} \\ A_{11}^{cT} & A_{12}^{cT} & A_{21}^{cT} & A_{22}^{cT} & \cdots & A_{l1}^{cT} & A_{l2}^{cT} \\ A_{11}^{kT} & A_{12}^{kT} & A_{21}^{kT} & A_{22}^{kT} & \cdots & A_{l1}^{kT} & A_{l2}^{kT} \end{bmatrix} \tag{11}$$

where A_{ir}^m, A_{ir}^c and $A_{ir}^k(i=1, 2, \cdots, l; r=1, 2)$ are submatrices which are composed of measured response data of the structure.

In general, the Eqn. (7) is an overdetermined linear simultaneous equation. The least squares solution of Eqn. (7) is determined by Gramm - Schmidt orthogonallizing method for its high accuracy of solution and good numerical stability[1].

3 PARAMETER OPTIMIZATION

The dynamical parameters of the model of a structure can, in theory, be determined accurately from measured response data of the structure by undetermined parameter method, but in fact it is imposible that there is not any error in the process of modelling. For this, there must be a certain error in the dynamical parameters to be calculated from response data. In order to improve the accuracy of modelling, or a good result is expected in case that there are large errors in measured response data, a parameter optimization procedure is presented in this paper.

3.1 Objective Function.

From modal analysis, one can know that the main parameters which express the model properties of a structure system are natural frequencies, damping ratios and vibrating modes of the system. However, for small damping structure, the natural frequencies and damping ratios can be obtained by more accurately testing. If the modal parameters of the model to be indentified are very close to that measured from the structure, then, this model can better express the dynamical characteristics of the system, the dynamical characteristics of the structure. For an N-degree-freedom system, the objective function is, therefore, taken as follows:

$$F(\{\boldsymbol{y}\})=\sum_{i=1}^{N} w_i|(\lambda_i-\overline{\lambda}_i)|^2 = \sum_{i=1}^{N}|u_i(\{\boldsymbol{y}\})|^2 \tag{12}$$

where $\{\boldsymbol{y}\}$ is a vector to be optimized, λ_i is the is eigenvalue of the system, $\overline{\lambda}_i$ is the measured eigenvalue of the structure, w_i is weighting coefficient which is used to blance the contribtion of each eigenvalue to the objective function $F(\{\boldsymbol{y}\})$ and is generally taken as $1/|\overline{\lambda}_i|$ or $1/|\overline{\lambda}_i|^2$.

Eqn. (12) can be put in the form as

$$F(\{\boldsymbol{y}\})=\{\delta^*\}^{\mathrm{T}}[\overline{w}]\{\delta\} \tag{13}$$

where $\{\delta\}=[\lambda_1-\overline{\lambda}_1, \lambda_2-\overline{\lambda}_2, \cdots, \lambda_N-\overline{\lambda}_N]^T$ $[\overline{w}]=[w_1, w_2, \cdots, w_N]$ and $\{\delta^*\}$ — conjugate transpose of vector $\{\delta\}$. Now, the problem is to find a new vector $\{\boldsymbol{y}\}$ which minimizes the objective function $F(\{\boldsymbol{y}\})$. For such an optimization problem, sequential unconstrained minimization technique (SUMT) or simplex method fails to give a satisfactory result and sometimes even leads to divergence. Therefore, the gradient search method is used for this problem. It shows that the gradient search method has faster convergence and needs less iteration time than other optimization methods for this problem.

3.2 Calculation of Jacobian Matrix

From $u_i(\{y\}) = \lambda_i - \bar{\lambda}_i$ we have

$$\begin{aligned} |u_i(\{y\})| &= |\lambda_i - \bar{\lambda}_i| \\ &= |(\lambda_{iR} - \bar{\lambda}_{iR}) + i(\lambda_{iI} - \bar{\lambda}_{iI})| \\ &= \sqrt{(\lambda_{iR} - \bar{\lambda}_{iR})^2 + (\lambda_{iI} - \bar{\lambda}_{iI})^2} \end{aligned} \tag{14}$$

In the process of parameter optimization, the gradiant of each function $|u_i(\{y\})|$, or Jacobian matrix

$$J = \begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1NV} \\ u_{21} & u_{22} & \cdots & u_{2NV} \\ u_{N1} & u_{N2} & \cdots & u_{NNV} \end{vmatrix} \tag{15}$$

will be used.

where

$$u_{ij} = \frac{\partial |u_i(\{y\})|}{\partial y_j} \cdot$$

$$\frac{(\lambda_{iR} - \bar{\lambda}_{iR})\dfrac{\partial \lambda_{iR}}{\partial y_j} + (\lambda_{iI} - \bar{\lambda}_{iI})\dfrac{\partial \lambda_{iI}}{\partial y_j}}{\sqrt{(\lambda_{iR} - \bar{\lambda}_{iR})^2 + (\lambda_{iI} - \bar{\lambda}_{iI})^2}} \tag{16}$$

$(i = 1, 2, \cdots, N,\ j = 1, 2, \cdots, NV)$

As can be seen from Eqn. (16), each element u_{ij} of matrix $[J]$ can be obtained so long as partial derivationes $\partial \lambda_{iR}/\partial y_j$ and $\partial \lambda_{iI}/\partial y_j$ have been calculated. For a system of N-degree of freedom with non-proportional viscous damping, the state vector equation can be written as follows:

$$[A]\dot{y} = -[B]y \tag{17}$$

where

$$[A] = \begin{bmatrix} C & M \\ M & O \end{bmatrix},$$

$$[B] = \begin{bmatrix} K & O \\ O & -M \end{bmatrix},$$

$$y = \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}$$

Thus, we have

$$\lambda_i [A]\{\Phi\}_i = -[B]\{\Phi\}_i \tag{18}$$

$$\lambda_i \{\Phi\}_i^T [A] = -[\Phi]_i^T [B] \tag{19}$$

where $\{\Phi\}_i$ —its eigenvector of the system. Differentiating Eqn. (18) with respect to y_j we can obtain

$$\frac{\partial \lambda_{iR}}{\partial y_j} = \frac{(a_1 c_1 + b_1 d_1)}{(c_1^2 + d_1^2)} \tag{20}$$

$$\frac{\partial \lambda_{iI}}{\partial y_j} = \frac{(b_1 c_1 - a_1 d_1)}{(c_1^2 + d_1^2)} \tag{21}$$

where

$$\begin{aligned} a_1 = & -\{\Phi\}_{iR}^T \frac{\partial [B]}{\partial y_j}\{\Phi\}_{iR} + \\ & \{\Phi\}_{iI}^T \frac{\partial [B]}{\partial y_j}\{\Phi\}_{iI} + \\ & \lambda_{iR}(-\{\Phi\}_{iR}^T \frac{\partial [A]}{\partial y_j}\{\Phi\}_{iR} + \\ & \{\Phi\}_{iI}^T \frac{\partial [A]}{\partial y_j}\{\Phi\}_{iI} + \\ & 2\lambda_{iI}\{\Phi\}_{iI}^T \frac{\partial [A]}{\partial y_j}\{\Phi\}_{iI} \\ b_1 = & -2\{\Phi\}_{iR}^T \frac{\partial [B]}{\partial y_j}\{\Phi\}_{iR} - \\ & 2\lambda_{iR}\{\Phi\}_{iI}^T \frac{\partial [A]}{\partial y_j}\{\Phi\}_{iI} + \\ & \lambda_{iI}(-\{\Phi\}_{iR}^T \frac{\partial [A]}{\partial y_j}\{\Phi\}_{iR} + \\ & \{\Phi\}_{iI}^T \frac{\partial [A]}{\partial y_j}\{\Phi\}_{iI} \\ c_1 = & \{\Phi\}_{iR}^T [A]\{\Phi\}_{iR} - \{\Phi\}_{iR}^T [A]\{\Phi\}_{iI} \\ d_1 = & 2\{\Phi\}_{iR}^T [A]\{\Phi\}_{iI} \end{aligned}$$

4 COMPUTER SIMULATION TEST

In order to examine the effectivenees of the technique proposed for the systems with different frequency bandwidth and different number of modes, three dynamical models of three degree of freedom and one of four degree-of-freedom were generated. The respronses of the system are computed using Eqn. (3). The amplitudes and phases of the responses with certain radom perturbations were inputted into program IDP and computed. The results calculated for model I are shown in Table 1 and Table 2 when the radom perturbation of the amplitudes was 15%, and the natural frequencies and damping ratios are shown in Table 3.

The results indicated that satisfactory identification of parameters can be obtained whether the coupling of system is strong or weak, and the frequency bandwidth is wide or narrow. The accuracy of parameters calculated is highly sensitive to the phase measured. If the parameters are not optimized, the error of parameters calculated are less than the corresponding perturbation of the amplitude. The accuracy of parameters calculated can be largely improved by parameter optimzation. From Talbe 1, it is shown that in case parameters are not optimized, the maximum relative error of parameter calculated is 15.27%, but after optimization the relative errors of the parameters calculated are all less than 5% except for one of them being 5.75%. As can be seen from Table 2 and Table 3, the accuracy of undamped natural mode of vibration, the natural frequency and damping ratios are raised apparently after optimizing.

5 MODELLING OF A DRILL

As an application, the dynamical model of Z512- 2 drill was established. A harmonic exciting force was applied at the top of the drill. The responses of point 2(driving piont) and point 4 are shown in Fig. 1 and Fig. 2 respectively. The responses of four points were measured as show in Fig. 3.

From the frequency response of the driving point, it shows that there are 4 main modes in the frequency range from 15 Hz to 400 Hz, which are 28.10, 140.11, 258.53 and 306.39 Hz separately. When the drill runs, the varying of axial force along the spindle is the major exciting source which causes the machine to vibrate, and the relative motion between spindle and working table affects processing accuracy. According to characteristics of the drill construction, the equivalent linear lumped parameter model of the drill is given as in Fig. 4.

As shown in Fig. 4, θ_1, θ_2, θ_3 and θ_4 are the coordinates of angular displacement; m_1, m_2,

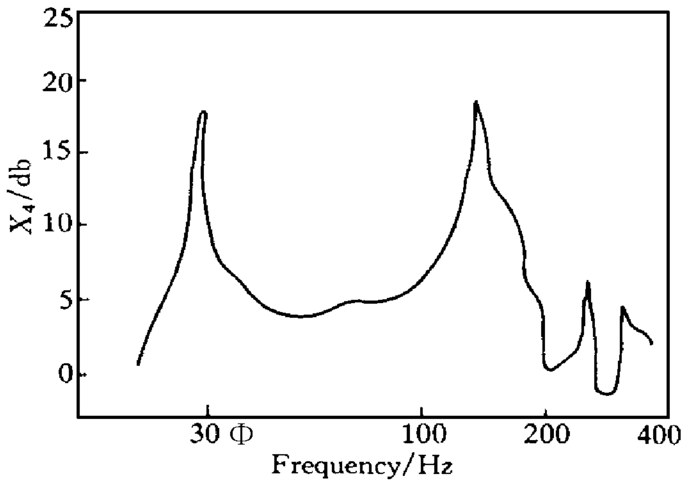

Fig. 1 Frequency response function of points 2

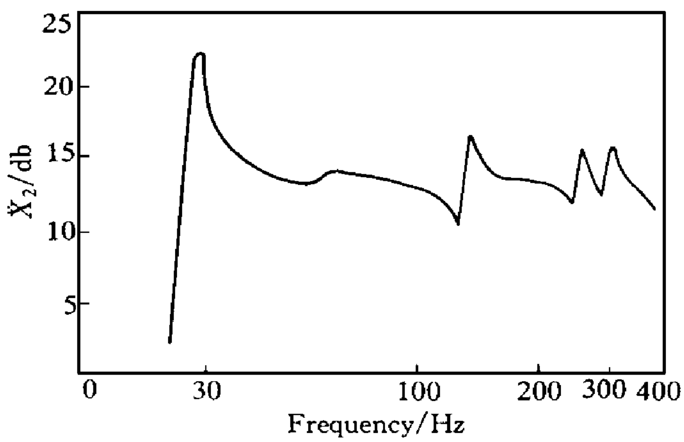

Fig. 2 Frequency response function of point 4

Table 1 The calculation results of matrices *[N], [C], [K]*

	Mass matrix [M]			Damping matrix[C]			Stiffness matrix[K]		
Original	0.800E1	0	0	0.450E2	- 0.450E2	0	0.900E5	- 0.900E5	0
	0	0.500E1	0	- 0.450E2	0.800E2	- 0.350E2	0.900E5	0.140E6	- 0.500E5
	0	0	0.900E1	0	- 0.359E2	0.750E2	0	- 0.500E5	0.120E6
Non-optimal	0.763E1	0	0	0.486E2	- 0.486E2	0	0.866E5	- 0.866E5	0
	0	0.491E1	0	- 0.486E2	0.805E2	- 0.319E2	- 866E5	0.129E6	- 0.424E5
	0	0	0.838E1	0	- 319E2	0.699E2	0	- 0.424E5	0.111E6
Optimal	0.763E1	0	0	0.436E2	- 0.436E2	0	0.886E5	- 886E5	0
	0	0.491e1	0	- 436E2	0.783E2	- 347E2	- 886E5	0.137E6	- 484E5
	0	0	0.838E1	0	- 0.347E2	0.708E2	0	- 0.484E5	0.113E6

Table 2 The calculation results of model matrix

	Original			Non-optimal			Optimal		
Modal matrix	0.2772	0.2332	0.1123	0.2849	0.2481	0.1056	0.2815	0.2387	0.1175
	0.1491	-0.0222	-0.3018	0.1404	-0.0200	-0.3180	0.1541	-0.2021	-0.3121
	0.1610	-0.3810	0.0863	0.1738	-0.3804	0.0838	0.1676	-0.3823	0.0898

Table 3 The natural frequencies and damping rations

	Frequency			Damping ratio		
	f_1	f_2	f_3	ζ_1	ζ_2	ζ_3
Original	0.6729E1	0.1890E2	0.3097E2	0.0130	0.0340	0.0540
Non-optimal	0.6595E1	0.1813E2	0.3029E2	0.0133	0.0348	0.0580
Optimal	0.6684E1	0.1823E2	0.3105E2	0.0129	0.0340	0.0543

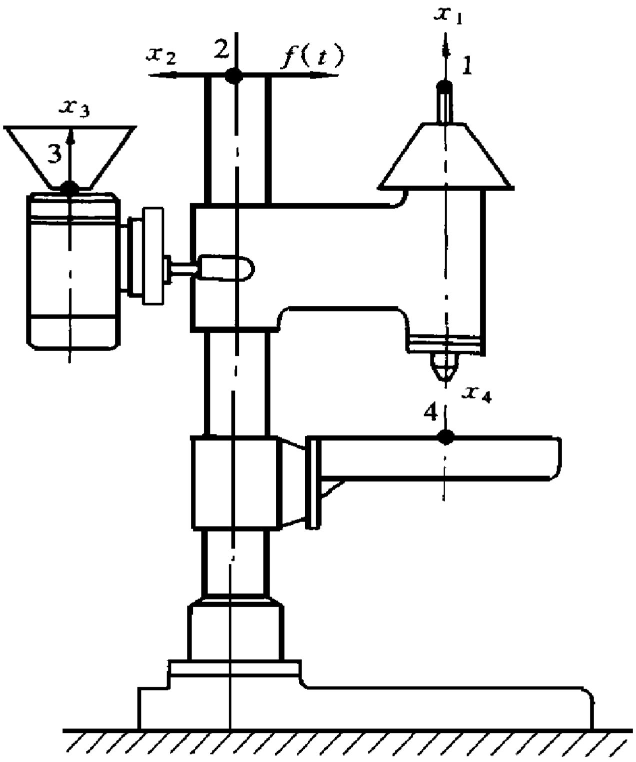

Fig. 3 The sketch of drill structure

● —measuring point

m_3 and m_4 are the equivalent masses on measuring points, respectively; m'_1, m'_2 and m'_4 are total masses of beams; k_1, k_2, k_3 and k_4 are the equivalent connection stiffness among structural parts; c_1, c_2, c_3 and c_4 are the equivalent damping coefficients.

According to Fig. 4, total kinetic energy of the drill can be written as follows:

$$T = \frac{1}{2}(I_1\dot{q}_1^2 + I_2\dot{q}_2^2 + I_3\dot{q}_3^2 + I_4\dot{q}_4^2) \tag{22}$$

Where q_i denotes the generalized coordinate in the present case, $q_i = \theta_i (i = 1, 2, \cdots, 4)$; while I_i denotes the moment of inertia of the drill structure part with respect to generalized coordinate q_i. The total potential energy of the system can be written as

$$U = \frac{1}{2}[k_1(q_1 - q_2)^2 + (k_2 q_2^2] + \frac{1}{2}[k_3(q_2 + q_3)^2 + k_4(q_4 - q_2)^2] \tag{23}$$

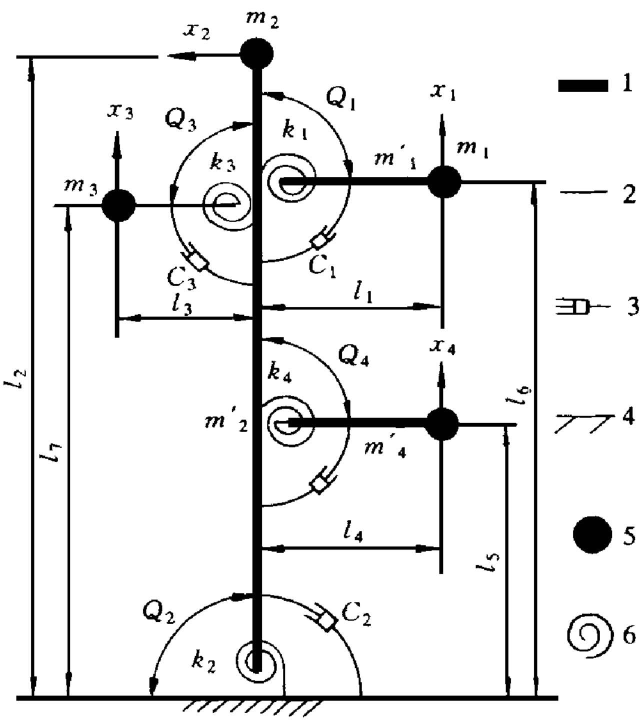

Fig. 4 mechanical model of drill

1 —Symbol distribated beam;
2 —rigid beam damper; 3 —rigid foundation;
4 —concentrated mass; 5 —twist spring

Table 4 Identified results for the drill

serial No. of equivalent mass	m_1	m'_1	m_2	m'_2	m_3	m_4	m'_4
value of mass/kg	4.74	5.16	11.35	28.64	2.91	8.57	5.18
Serial No. of damper		c_1		c_2	c_3		c_4
damping value NM/rad·s^{-1}		0.1901E2		0.1970E2	0.2906E2		0.8665E2
serial No. of equivalent spring conatant		k_1		k_2	k_3		k_4
value of sparing constant(NM/rad)		0.1140E6		0.5593E6	0.2753E6		0.1359E7

The generalized forces acted on the system including:

conservative force: $-\dfrac{\partial U}{\partial q_i}$, $(i = 1, 2, 3, 4)$

moment of damping force: $\sum_{i=1}^{n} f_{1i}$, $(i = 1, 2, 3, 4)$

moment of exciting force: $F_i 1_i$, $(i = 1, 2, 3, 4)$

By applying Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + \sum_{l=1}^{n} f_{1i} + F_i l_i \tag{24}$$

the dynamic equation of the structure system can be derived:

$$\mathrm{T}\begin{bmatrix} I_1/l_1^2 & 0 & 0 & 0 \\ 0 & I_2/l_2^2 & 0 & 0 \\ 0 & 0 & I_3/l_3^2 & 0 \\ 0 & 0 & 0 & I_4/l_4^2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} c_1/l_1^2 & -c_1/l_1 l_2 & 0 & 0 \\ -c_1/l_1 l_2 & c_2/l_2^2 & c_3/l_3 & -c_4/l_2 l_4 \\ 0 & c_3/l_2 l_3 & c_3/l_3^2 & 0 \\ 0 & -c_4/l_2 l_4 & 0 & c_4/l_4^2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}$$
$$\begin{bmatrix} k_1/l_1^2 & -k_1/l_1 l_2 & 0 & 0 \\ -k_1/l_1 l_2 & k/l_2^2 & k_3/l_2 l_3 & -k_4/l_2 l_4 \\ 0 & k_3/l_2 l_3 & k_3/l_3^2 & 0 \\ 0 & -k_4/l_2 l_4 & 0 & k_4/l^2 l_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \tag{25}$$

where $c = c_1 + c_2 + c_3 + c_4$, $k = k_1 + k_2 + k_3 + k_4$.

After the dynamic equation (25) of the structure system is obtained, the structure dynamic parameters can be solved from the measured discrete response data of the structure at every sampling frequency.

An example of practical case of application, dynamic modelling of a drill is pressented. The identified results of the drill are shown in Talbe 4. Thus the dynamic model of the drill can be finally established.

6 CONCLUSION

The mathematical modelling of a structure and its dynamical modelling are taken together in this paper. It not only can ensure that the mathematical modelling is in accordance with the dynamical modelling of the structure, but also can simplify the calculation by using some characteristics of mass, damping and stiffnes matrices, so that the accuracy of the modelling of structure can be raised.

The parameter of the model not only used in modelling, but also in designing of model so that its natural characteristies should be satisfactory to some requirements given.

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