OPTIMIZATION OF CUT-OFF GRADE IN OPEN-PIT BASED ON CONTROL THEORY[①]

Xie Yingliang
Department of Management,
South Institute of Metallurgy, Ganzhou 341000, P. R. China

ABSTRACT A new method for the optimization of cut-off grade in open-pit was put forward, which was based on control theory. After a brief introduction to the analytical framework of calculus of variations with an overall equality constraint, a mathematical model for selecting an optimum cut-off grade function was constructed, which aimed at the maximization of total present value of an open-pit. Then a comparison was made, and it was shown that the new method yielded the same solution results as the traditional ones, and in some cases it avoided the iteration process which was needed by those traditional methods due to the mutural determination of cut-off grade and the maximum total present value of a mine. At last a calculation example was given.

Key words cut-off grade open-pit control theory

1 INTRODUCTION

In an open-pit mine the cut-off grade is an operation control, used as a guideline to seperate ore and waste. The purpose of the calculation of optimum cut-off grades is to maximize a mining firm′s total present value. Over the last three decades, great progress has been made in the study of the economics of cut-off grade [1-3]. The application of modern economic theories and concepts such as marginal analysis and opportunity cost resulted in a sound understanding of the subject among both mineral economists and industrialists, and having a declining cut-off grade series for maximization was widely accepted and applied in mining practice. A number of mathematical models have been built for the calculation in which an iteration process was involved due to the mutural determination of optimum cut-off grade and the maximum present value of a mine. A new method based on control theory, or more specifically, on calculus of variations was put forward here. This method produced the same solution results as the existing methods and in some cases, for an exponential distribution of grade, for example, the iteration process in the calculation was avoided. Besides, it may provide a new tool for the modeling of more complicated cut-off grade studies such as problems with cyclical metal price fluctuations within mine life.

2 ANALYTICAL MODEL OF CALCULUS OF VARIATIONS[4]

A classical model of optimization by calculus of variations is to select $y^* = f^*(x)$ from $y = f(x)$ to maximize or minimize the objective functional function $I(y)$

$$I(y) = \int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x$$

with the boundary conditions of $y(x_0) = y_0$ and $y(x_1) = y_1$. The necessary condition of a maximization or minimization is determined by Euler equation:

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'} = 0 \qquad (1)$$

where F_y and $F_{y'}$ are the calculus of variations of F concerning y and y' respectively. For a maximization with an overall equality constraint, i. e., a constraint in a form as

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$$\int_{x_0}^{x_1} G(x, y, y')\mathrm{d}x = K$$

where K is a constraint constant, we introduce a Lagrange's function L,

$$L(x, y, y', \lambda) = F(x, y, y') - \lambda G(x, y, y')$$

where λ is a Lagranges multiplier, and replaces F in Euler equation with L, then the necessary condition for maximization is obtained:

$$Ly - \frac{\mathrm{d}}{\mathrm{d}x}Ly' = 0 \qquad (2)$$

where Ly and Ly' are the calculus of variations of L concerning y and y' respectively.

3 MATHEMATICAL MODEL OF OPTIMUM CUT-OFF GRADE BASED ON CONTROL THEORY

The commonly accepted dynamic analysis suggests that the cut-off grade, α, is a function of residual mine life, hence a function of time, i. e. $\alpha = h(t)$. Therefore according to control theory, the study of cut-off grade is to select $\alpha^* = h^*(t)$ from $\alpha = h(t)$ to maximize the total present value of a mine under the constraint of given mineralized material within the pit limit.

Letting R and C be respectively the revenue and the cash cost per unit of time (per year in this paper), then, $R = Q\varepsilon\bar{\alpha}K$ and

$$C = Q\frac{m}{p} + cQ + F_0$$

where Q —Concentrator feed rate, ε—Fraction of mineral in ore feed that is recovered in concentration, K —Net value per unit mineral contained in concentrate products, m —Variable mining cost per unit of material, c —Variable concentrating cost per unit of ore, F_0 —Fixed cash cost per unit of time; p —Fraction of mined material which is of higher grade than the current cut-off grade, α, and then classed as ore, and it is obtained as

$$p = \int_{\alpha}^{1} g(z)\mathrm{d}z$$

$\bar{\alpha}$ —Average grade of selected ore, it is given by

$$\bar{\alpha} = \frac{1}{p}\int_{\alpha}^{1} zg(z)\mathrm{d}z$$

where $g(z)$ is the function of distribution density of ore body. Then the present value of the net cash flow υ at time t, is given by

$$\upsilon = [Q\varepsilon\bar{\alpha}K - (Q\frac{m}{p} + cQ + F_0)]\mathrm{e}^{-i(t_1 - t)}$$

where i —Discount rate, t_1 —Residual life of a mine at the present time. Therefore, the total present value of the mine, V, is

$$V(\alpha) = \int_{t_0}^{t_1} v\,\mathrm{d}t \qquad (3)$$

Since the mined material required for the feeding of Q at the cut-off grade is Q/p, the overall equality constraint is

$$\int_{t_0}^{t_1} \frac{Q}{p}\mathrm{d}t = Q_0 \qquad (4)$$

where Q_0 —Amount of mineral-bearing material remaining within pit limits, on which the estimation of distribrtion density is based; t_0 —Time point at the end of mine life, hence $t_0 = 0$.

For the boundary conditions, $h(t_1)$ is determined by the constraint equation, and $h(t_0)$, i. e. the cut-off grade at the end of the mine's operation, is given by break-even analysis which yields the lowest level of cut-off grade for a mine. According to Refs. [3] and [5], then

$$h(t_0) = \frac{c + F_0/Q}{\varepsilon K}$$

Let $F = v$, and $G = Q/p$, a Lagrangeian function was constructed:

$$L(t, \alpha, \alpha') = F(t, \alpha, \alpha') - \lambda G(t, \alpha, \alpha')$$

According to Eqn. (2), it is obtained as

$$L_\alpha - \frac{\mathrm{d}}{\mathrm{d}x}L_{\alpha'} = 0 \qquad (5)$$

where $L_{\alpha'} = 0$ and $L_\alpha = F_\alpha - \lambda G_\alpha$

$$F_\alpha = [Q\varepsilon K F_{\bar{\alpha}}\frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}\alpha} - mQF_p\frac{\mathrm{d}p}{\mathrm{d}\alpha}]\mathrm{e}^{-i(t_1 - t)}$$

$$= [Q\varepsilon K\frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}\alpha} + \frac{mQ}{p^2}\frac{\mathrm{d}p}{\mathrm{d}\alpha}]\mathrm{e}^{-i(t_1 - t)}$$

$$G_\alpha = QG_p\frac{\mathrm{d}p}{\mathrm{d}\alpha} = -\frac{Q}{p^2}\frac{\mathrm{d}p}{\mathrm{d}\alpha}$$

Applying F_α and G_α to Eqn. (5), then

$$[Q\varepsilon K\frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}\alpha} + \frac{mQ}{p^2}\cdot\frac{\mathrm{d}p}{\mathrm{d}\alpha}]\mathrm{e}^{-i(t_1 - t)} + \lambda\frac{Q}{p^2}\cdot\frac{\mathrm{d}p}{\mathrm{d}\alpha} = 0 \qquad (6)$$

This equation, together with the boundary condition $h(t_0)$ and the constraint equation, gives the solution of optimum cut-off grade α^*.

For an exponential grade distribution density, i. e. $g(z) = b e^{a-bz}$, one has

$$p = \int_{\alpha}^{1} b e^{a-bz} \mathrm{d}z = e^{a-b\alpha} \qquad (7)$$

and

$$\bar{\alpha} = \frac{1}{e^{a-b\alpha}} \int_{\alpha}^{1} z b e^{a-bz} \mathrm{d}z = \alpha + \frac{1}{b} \qquad (8)$$

where a and b are constants of the distribution density.

The combination of Eqns. (7), (8) and (6) results in

$$(Q\varepsilon K - Qbm e^{b\alpha-a}) e^{-i(t_1-t)} = \lambda Q b e^{b\alpha-a}$$

After some rearrangements, it is yielded

$$e^{b\alpha-a} = \frac{\varepsilon K e^{-i(t_1-t)}}{b[m e^{-i(t_1-t)} + \lambda]}$$

from which the optimum cut-off grade function concerning time, α^* is obtained,

$$\alpha^* = \frac{1}{b} \ln \frac{\varepsilon K e^{-i(t_1-t)}}{b[m e^{-i(t_1-t)} + \lambda]} + \frac{a}{b} \qquad (9)$$

where λ and t_1 are derived through Eqns. (10) and (11) as follows.

Using the boundary condition of $t_0 = 0$ and $h(t_0)$, then from Eqn. (9), there is

$$\frac{c + F_0/Q}{\varepsilon K} = \frac{1}{b} \ln \frac{\varepsilon K e^{-it_1}}{b[m e^{-it_1} + \lambda]} + \frac{a}{b} \qquad (10)$$

By means of the constraint equation, then

$$Q_0 = \int_{t_0}^{t_1} \frac{Q}{p} \mathrm{d}t = \int_{t_0}^{t_1} Q e^{b\alpha-a} \mathrm{d}t$$

Applying α^* to the above equation, and after rearrangement, it is obtained

$$Q_0 = \int_{t_0}^{t_1} Q \frac{\varepsilon K e^{-i(t_1-t)}}{b[m e^{-i(t_1-t)} + \lambda]} \mathrm{d}t$$

This equation results in

$$\frac{Q\varepsilon K}{mib} \ln \frac{m + \lambda}{m e^{-it_1} + \lambda} = Q_0 \qquad (11)$$

When $Q_0 \to +\infty$, this means that there is no constraint on the quantity of mineralized material of a mine and then $t_2 \to \infty$ and $\lambda \to 0$. In this case, from Eqn. (9), it is obtained

$$\frac{1}{b} e^{a-b\alpha} = \frac{m}{\varepsilon K}$$

Considering $\frac{1}{b} = \bar{\alpha} - \alpha$, and $e^{a-b\alpha} = p$, then

$$p(\bar{\alpha} - \alpha) = \frac{m}{\varepsilon K} \qquad (12)$$

Eqn. (12) is exactly the same one as Eqn. (9) obtained by Schaap in Ref. [3].

Or when $i \to +\infty$, this means that the mining firm places the paramount priority to the immediate profit, and then from Eqn. (9), Eqn. (12) can also be got.

More generally, since

$$\frac{\mathrm{d}p}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha} \int_{\alpha}^{1} g(z) \mathrm{d}z = -g(z) \quad \text{and}$$

$$\frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha} \left[\frac{1}{p} \int_{\alpha}^{1} z g(z) \mathrm{d}z\right] = (\bar{\alpha} - \alpha) \frac{g(z)}{p}$$

applying $\frac{\mathrm{d}p}{\mathrm{d}\alpha}$ and $\frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}\alpha}$ to Eqn. 6, and letting $\lambda = 0$, then

$$\left[Q\varepsilon K(\bar{\alpha} - \alpha) \frac{g(z)}{p} - \frac{mQ}{p^2} \cdot g(z)\right] e^{-i(t_1-t)} = 0$$

From this equation, Eqn. (12) is got. This indicates the consistency of the new method with the traditional method based on marginal analysis in terms of the two extreme operating points.

4 A CALCULATION EXAMPLE

From a hypothetical metal mine, the following data are obtained:

$Q_0 = 40 \times 10^6$ t, $Q = 1.5 \times 10^6$ t/a,

$F_0 = 19.5 \times 10^6$ Yuan/a,

$K = 6,000$ Yuan/t

$m = 11$ Yuan/t, $c = 11$ Yuan/t,

$\varepsilon = 95\%$

The distribution density is supposed to be exponential, with $a = 1.2$ and $b = 300$.

Applying these data to Eqn. (9), (10) and (11), then

$$\alpha^* = \frac{1}{300} \ln \frac{5\,700 e^{-0.1(t_1-t)}}{3\,300 e^{-0.1(t_1-t)} + 300\lambda} + \frac{1.2}{300} \qquad (13)$$

$$\ln \frac{5\,700 e^{-0.1t_1}}{3\,300 e^{-0.1t_1} + 300\lambda} = 0.06 \qquad (14)$$

and

$$\ln \frac{11 + \lambda}{11 e^{-0.1t} + \lambda} = 1.54 \qquad (15)$$

The solution of Eqns. (14) and (15) yields: $t_1 = 19.35$ and $\lambda = 1$. Applying the solution of t_1 and λ to Eqn. 13, it is obtained

$$\alpha^* = \frac{1}{300} \ln \frac{5\,700 e^{-0.1(19.35-t)}}{3\,300 e^{-0.1(19.35-t)} + 300} + \frac{1.2}{300} \qquad (16)$$

Letting $t = 0, 1, \cdots, 19.35$, from Eqn. (16) with a discount rate of 10%, the obtained solution results of optimum cut-off grade are as shown in Table 1.

Table 1 shows that the mine life of 19.35 a needs 41.6×10^6t of mineralized material, which exceeds the constraint of 40×10^6t. Therefore the practical mine life is 18+ (40.78− 40)/2.36 = 18.67. Slight difference exists between 18.67 a mine life in Table 1 and that obtained from Eqns. (14) and (15) which is 19.35 a. This is because the model demands a continuous change of cut-off grade while in practice only annual adjustment is carried out. As a calculation experiment, if a semi-annual adjustment is implemented, the mine life extends to 19.1a, which is much closer to the optimum one. The annual (or semi-annual) change in cut-off grade can be seen as a second best policy for a mine.

Table 1 Optimum cut-off grade and relating variables

t	α^* /%	p	$\bar{\alpha}$ /%	$\frac{Q}{p}$ /10^6t	$\Sigma(\frac{Q}{p})$ /10^6t
0	0.419	0.943	0.752	1.59	1.59
1	0.432	0908	0.762	1.65	3.24
2	0.444	0.876	0.777	1.71	4.95
3	0.455	0.848	0.788	1.76	6.71
4	0.465	0.823	0.798	1.82	8.53
5	0.474	0.801	0.807	1.87	10.40
6	0.483	0.780	0.816	1.92	12.32
7	0.492	0.759	0.825	1.97	14.29
8	0.499	0.743	0.832	2.01	16.30
9	0.506	0.728	0.839	2.06	18.36
10	0.513	0.712	0.846	2.10	20.46
11	0.519	0.700	0.852	2.14	22.60
12	0.524	0.689	0.857	2.17	24.77
13	0.529	0.679	0.862	2.20	26.97
14	0.534	0.669	0.867	2.24	29.21
15	0.538	0.661	0.871	2.26	31.47
16	0.532	0.653	0.875	2.29	33.76
17	0.546	0.645	0.879	2.32	36.08
18	0.549	0.640	0.882	2.34	38.42
19	0.552	0.634	0.885	2.36	40.78
19.35	0.553	0.632	0.886	0.82	41.60

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