EIGEN THEORY OF ELASTIC DYNAMICS FOR ANISOTROPIC SOLIDS①

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ABSTRACT The process of anisotropic elastic dynamics was considered under the standard space, and the corresponding eigen equations were obtained. They are independent of each other and represent various types of waves respectively. The mixed wave of anisotropic elastic body consists of them. Following results were given: (1) the number of elastic waves of anisotropic body is equal to that of the subspaces of anisotropy; (2) the propagating speed of each elastic wave is only related to the eigen elasticity and the space module of the corresponding subspace; (3) the type of elastic waves depends on the mechanical meaning of modal strain under the standard space. Finally, the properties of elastic waves of various crystals were discussed.

Key words anisotropic body elastic wave standard space eigen dynamic equation

1 INTRODUCTION

The concept of eigen elasticity originated from Kelvin's work[1,2]. It is in last decade that eigen elasticity was studied again and developed, which brought about the concept of standard space[3-5]. Based on these, a new theory on constitution and evolution of anisotropic body was presented[6-8], from which, people can easily study the complicated problems of anisotropic body. In this paper, the author make a new exploration in the field of dynamics. It is well known that there is a great difficulty in solving the problems of anisotropic dynamics because of a large amount of independent elastic coefficients. Only for a few crystals, such as cubical crystal, can the solutions of plane wave on the spcial plane be given in the form of the longitudinal wave and transverse wave. For other more complicated crystals, there are no solutions at all. For example, up to now, we still don't know the relationships between the number of elastic waves and anisotropy, between the propagating speed of elastic waves and elastic coefficients and between the mixed wave and decomposed waves, and so on. However, the theory given here investigates the elastic dynamics of anisotropic body under the anisotropic subspace, so we can get the equations of the spectral form of elastic dynamics, which are independent of each other because of the orthogonality of the subspaces. Thus, all of the informations on elastic waves of anisotropic body can be obtained completely. It is very important to engineering[9,10].

2 CONCEPT OF STANDARD SPACE[3-5]

The eigenvalue problem of elastic mechanics can be written as follows:

$$\boldsymbol{C}\varphi_i = \lambda_i \varphi_i \quad (i = 1, 2, \cdots, 6) \tag{1}$$

$$\boldsymbol{C}^{-1}\varphi_i = \mu_i \varphi_i \quad (i = 1, 2, \cdots, 6) \tag{2}$$

where $\boldsymbol{C}$ is a matrix of the elastic coefficients, λ_i and μ_i are eigen elasticity and eigen flexibility respectively, and are invariables of coordinates. φ_i is the corresponding eigen vector, and meets the condition of orthogonality.

The anisotropic subspace of elastic body consists of the independent eigen vectors, that is

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the standard space.

$$W = W_1[\varphi_1^*] \oplus \cdots \oplus W_M[\varphi_M^*] \quad (3)$$

where the possible overlapping roots are considered, and using $M(\leqslant 6)$ represents the number of anisotropic subspaces. Because the anisotropic subspaces are split out of the elastic coefficients, projecting the stress vector and strain vector on the subspaces. We have:

$$\sigma = \sigma_1^* \varphi_1^* + \cdots + \sigma_M^* \varphi_M^* \quad (4)$$

$$\boldsymbol{e} = e_1^* \varphi_1^* + \cdots + e_M^* \varphi_M^* \quad (5)$$

where σ_i^* and e_i^* are stress and strain under the standard space respectively, they are different from the previous ones in the mechanical meaning, and are often called as the modal stress and the modal strain. Eqns.(4) and (5) are also regarded as a result of various modal sum. The modal stress and strain hold Hook's law:

$$\sigma_i^* = \lambda_i e_i^* \quad (i = 1, 2, \cdots, M) \quad (6)$$

3 EIGEN ELASTIC WAVE EQUATIONS OF ANISOTROPIC BODY

When neglecting body force, the dynamic equation and geometric equation of elastic body are respectively:

$$\sigma_{ik'k} = \rho \ddot{u}_i \quad (7)$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i'j} + u_{j'i}) \quad (8)$$

From them, we can get the following equation

$$\sigma_{ik'kj} + \sigma_{jk'ki} = 2\rho\ddot{\varepsilon}_{ij} \quad (9)$$

Because of the symmetry on (i, j) in Eqn. (9), we can rewrite it in the form of vector, and substitute the engineering strain for the strain tensor, we have:

$$\Delta \sigma = \rho \Delta_{tt} \boldsymbol{e} \quad (10)$$

where

$$\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}]^{\mathrm{T}},$$

$$\boldsymbol{e} = [e_{11}, e_{22}, e_{33}, e_{23}, e_{13}, e_{12}]^{\mathrm{T}},$$

Δ is a symmetrical differential operator matrix, and $\partial_{ij} = \partial_{ji} = \partial^2/\partial x_i \partial x_j$, $\Delta_{tt} = \partial^2/\partial t \partial t$.

$$\Delta = \begin{bmatrix} \partial_{11} & 0 & 0 & 0 & \partial_{31} & \partial_{21} \\ 0 & \partial_{22} & 0 & \partial_{32} & 0 & \partial_{21} \\ 0 & 0 & \partial_{33} & \partial_{32} & \partial_{31} & 0 \\ 0 & \partial_{23} & \partial_{23} & (\partial_{22} + \partial_{33}) & \partial_{21} & \partial_{31} \\ \partial_{13} & 0 & \partial_{13} & \partial_{12} & (\partial_{11} + \partial_{33}) & \partial_{32} \\ \partial_{12} & \partial_{12} & 0 & \partial_{13} & \partial_{23} & (\partial_{22} + \partial_{11}) \end{bmatrix} \quad (11)$$

Appendix A proves that the elastic dynamic Eqn. (10) under the geometric space is of following eigen form under the standard space.

$$\eta_i \lambda_i e_i^* = \rho \Delta_{tt} e_i^* \quad (i = 1, 2, \cdots, M) \quad (12)$$

and

$$\eta_i = \varphi_i^{*\mathrm{T}} \Delta \varphi_i^* \quad (i = 1, 2, \cdots, M) \quad (13)$$

where η_i is an eigen differential operator, φ_i^* can be regarded as the restrictive vector of anisotropic elasticity on dynamic equation. The calculation shows that the eigen differential operator is the same as Laplace operator (either two dimensions or three dimensions) for isotropic body, and for most of anisotropic bodies. Therefore, the eigen elastic wave equations under the standard space are of the following common form:

$$\lambda_i \beta_i \nabla_i^2 e_i(x, t) = \rho \ddot{e}_i(x, t) \quad (i = 1, 2, \cdots, M) \quad (14)$$

where ∇_i^2 is Laplace operator of order i, β_i is the modulo of standard space of order i. Eqn. (14) shows that the number of elastic waves of anisotropic body is equal to that of the subspaces of anisotropy. Therefore the propagating speed of elastic waves under its subspace can be calculated as the following equation:

$$v_i = \sqrt{\beta_i \frac{\lambda_i}{\rho}} \quad (i = 1, 2, \cdots, M) \quad (15)$$

4 ELASTIC WAVE IN ISOTROPIC BODY

There are two independent anisotropic subspaces in an isotropic body:

$$W = W_1^{(1)}[\varphi_1] \oplus W_2^{(5)}[\varphi_2, \cdots, \varphi_6] \quad (16)$$

where

$$\left.\begin{aligned} \varphi_1 &= \frac{\sqrt{3}}{3}[1,1,1,0,0,0]^T \\ \varphi_2 &= \frac{\sqrt{2}}{2}[0,1,-1,0,0,0]^T \\ \varphi_3 &= \frac{\sqrt{6}}{6}[2,-1,-1,0,0,0]^T \\ \varphi_i &= \xi_i \quad (i = 4,5,6) \end{aligned}\right\} \quad (17)$$

and ξ_i is a vector of order 6 in which ith element is 1 and others are 0.

The corresponding eigen elasticity and eigen differential operators are:

$$\left.\begin{aligned} \lambda_1 &= 3(\lambda + 2\mu), \lambda_2 = 2\mu \\ \eta_1 &= \frac{1}{3}\nabla^2_{III}, \eta_2 = \frac{1}{2}\nabla^2_{III} \end{aligned}\right\} \quad (18)$$

where ∇^2_{III} is Laplace operator of tree dimension.

Thus, there exist two independent elastic waves in isotropic body. They obey the following equations:

$$(\lambda + 2\mu)\nabla^2_{III} e_1^*(x,t) = \rho\ddot{e}_1^*(x,t) \quad (19)$$

$$\mu\nabla^2_{III} e_2^*(x,t) = \rho\ddot{e}_2^*(x,t) \quad (20)$$

It will be seen as follows that Eqns.(19) and (20) represent the dilatant and shear wave respectively.

From Eqn.(5), the modal strain of order 1 is:

$$e_1^* = \varphi_1^{*T} \cdot \boldsymbol{e} = \frac{\sqrt{3}}{3}(e_{11} + e_{22} + e_{33}) \quad (21)$$

Eqn.(21) represents the relative change of the volume of elastic body. So, Eqn.(19) shows the motion of pure longitudinal wave.

Also from Eqn.(5), the modal strain of order 2 is:

$$e_2^* \varphi_2^* = \boldsymbol{e} - e_1^* \varphi_1^* \quad (22)$$

Using the condition of orthogonality, than we have:

$$\begin{aligned} |e_2^*| &= [(\boldsymbol{e} - e_1^* \varphi_1^*)^T \cdot (\boldsymbol{e} - e_1^* \varphi_1^*)]^{1/2} \\ &= \{\frac{1}{3}[(e_1 - e_2)^2 + (e_2 - e_3)^2 + (e_3 - e_1)^2]\}^{1/2} \end{aligned} \quad (23)$$

Eqn.(23) represents the pure shear strain on the eight-side body. So Eqn.(20) shows the motion of pure transverse wave.

5 ELASTIC WAVE IN ANISOTROPIC BODIES

5.1 Cubical crystal

$$W = W_1^{(1)}[\varphi_1] \oplus W_2^{(2)}[\varphi_2, \varphi_3] \oplus W_3^{(3)}[\varphi_4, \varphi_5, \varphi_6] \quad (24)$$

where $\varphi_1, \varphi_2, \cdots, \varphi_6$ are the same as in Eqn.(17).

$$\left.\begin{aligned} \lambda_1 &= c_{11} + 2c_{12} \\ \lambda_2 &= c_{11} - c_{12} \\ \lambda_3 &= c_{44} \\ \beta_1 &= \frac{1}{3} \\ \beta_2 &= \frac{1}{2} \\ \beta_3 &= 1 \end{aligned}\right\} \quad (25)$$

$$\lambda_i\beta_i \nabla^2_{III} e_i^*(x,t) = \rho\ddot{e}_i^*(x,t) \quad (i = 1,2,3) \quad (26)$$

$$e_1^* = \frac{\sqrt{3}}{3}(e_{11} + e_{22} + e_{33}) \quad (27)$$

$$e_2^* = [\frac{1}{2}(e_{22} - e_{33})^2 + \frac{1}{6}(2e_{11} - e_{22} - e_{33})^2]^{1/2} \quad (28)$$

$$e_3^* = \sqrt{\frac{1}{2}(e_{32}^2 - e_{31}^2 - e_{12}^2)} \quad (29)$$

There exist three elastic waves in cubical crystal, one is the dilatant wave and two others are shear waves.

5.2 Six-angle crystal

$$W = W_1^{(1)}[\varphi_1] \oplus W_2^{(1)}[\varphi_2] \oplus W_3^{(2)}[\varphi_3, \varphi_6] \oplus W_4^{(2)}[\varphi_4, \varphi_5] \quad (30)$$

where

$$\left.\begin{aligned} \varphi_{1,2} &= \frac{c_{13}}{\sqrt{(\lambda_{1,2} - c_{11} - c_{12})^2 + 2c_{13}^2}} \times [1,1,\frac{\lambda_{1,2} - c_{11} - c_{12}}{c_{13}},0,0,0]^T \\ \varphi_3 &= \frac{\sqrt{2}}{2}[1,-1,0,0,0,0]^T \\ \varphi_i &= \xi_i, \ i = 4,5,6 \end{aligned}\right\} \quad (31)$$

$$\left.\begin{aligned}\lambda_{1,2} &= \frac{c_{11}+c_{12}+c_{33}}{2} + \sqrt{\left(\frac{c_{11}+c_{12}-c_{33}}{2}\right)^2 + 2c_{13}^2}\\ \lambda_3 &= c_{11}-c_{12},\ \lambda_4 = c_{44}\\ \beta_{1,2} &= \frac{c_{13}^2}{(\lambda_{1,2}-c_{11}-c_{12})^2+2c_{13}^2}\\ \beta_3 &= \frac{2}{3},\ \beta_4 = \frac{1}{2}\end{aligned}\right\} \quad (32)$$

$$\lambda_i\beta_i \nabla^2_{\mathrm{III}} e_i^*(x,t) = \rho\ddot{e}_i^*(x,t) \quad (i=1,2) \quad (33)$$

$$\lambda_3\beta_3 \nabla^2_{\mathrm{II}} e_3^*(x,t) = \rho\ddot{e}_3^*(x,t) \quad (34)$$

$$\lambda_4\beta_4(\nabla^2_{\mathrm{III}} + 2\partial_{12})e_4^*(x,t) = \rho\ddot{e}_4^*(x,t) \quad (35)$$

where ∇^*_{II} is Laplace operator of two dimension.

$$e_{1,2}^* = \frac{c_{13}}{\sqrt{(\lambda_{1,2}-c_{11}-c_{12})^2+2c_{13}^2}} \times \left[e_{11}+e_{22}+\left(\frac{\lambda_{1,2}-c_{11}-c_{12}}{c_{13}}\right)e_{33}\right] \quad (36)$$

$$e_3^* = \sqrt{\frac{1}{2}(e_{11}+e_{22})^2 + e_{12}^2} \quad (37)$$

$$e_4^* = \sqrt{\frac{1}{2}\left[e_{32}^2 + e_{31}^2\right]} \quad (38)$$

There exist four elastic waves in six-angle crystal. Two are dilatant waves and two others are shear waves, in which third wave is a plane wave.

6 CONCLUSIONS

Using the way of projecting the mechanical qualities on the standard space rather than the geometrical space, the elastic dynamic equation is divided into several eigen dynamic equations, from which we obtain a general formula of anisotropic elastic waves and prove the following results:

(1) The number of elastic waves of anisotropic body is equal to that of anisotropic subspaces;

(2) The propagating speed of elastic waves of anisotropic body is directly proportional to the eigen elasticity and module of the corresponding anisotropic subspace;

(3) The propagating form of elastic waves of anisotropic body is dependent on the mechanical meaning of the modal strain. For example, if the modal strain represents the change of volume, the elastic wave is dilatant type, and if the modal strain represents the change of shape, the elastic wave is shear type;

(4) The mixed wave of anisotropic body is the modal sum of different elastic waves.

The calculation shows that the propagating forms of elastic waves in crystals are all incomplete dilatant type or incomplete shear type except the pure longitudinal or pure transverse waves in an isotropic body.

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Appendix A
Proof of Eigen Elastic Dynamics

The generalized Hooke's law and dynamical equation are respectively

$$\sigma = \boldsymbol{Ce} \quad (\mathrm{A1})$$

$$\Delta\sigma = \rho\Delta_{tt}\boldsymbol{e} \quad (\mathrm{A2})$$

Let $\sigma = \lambda\alpha\varphi$, in which α is a time-space variable, φ is an unknown vector, and λ is an unknown constant. According to the generalized Hooke's law, if $\boldsymbol{e} = \alpha\varphi$, λ and φ must be nonzero solutions of following equation:

$$(C - \lambda I)\varphi = 0 \quad (A3)$$

It is seen that λ and φ are eigenvalue and eigenvector of the elastic coefficient matrix C. So we get the following equations:

$$C\varphi_i = \lambda_i\varphi_i \quad (i = 1, 2, \cdots, 6) \quad (A4)$$

$$C\Phi = \Phi\Lambda \quad (A5)$$

$$C = \Phi\Lambda\Phi^{-1} = \Phi\Lambda\Phi^{T} \quad (A6)$$

where Λ is eigen elastic moduli matrix, which is the ciagonal matrix, Φ is the modal matrix, which is the orthogonal one.

Substituting the stress vector $\sigma = \lambda\alpha\varphi$ and strain vector $e = \alpha\varphi$ into the dynamical Eqn. (A2), we have

$$\Delta(\alpha\varphi) = \frac{\rho\Delta_{tt}}{\lambda}(\alpha\varphi) \quad (A7)$$

It is seen from Eqn.(A7) that under the condition of elasticity, the geometric differential operator matrix Δ in the dynamical equation also has the eigen properties, and the following equation is held.

$$(\Delta - \eta I)(\alpha\varphi) = 0 \quad (A8)$$

Transposing Eqn.(A8), We have:

$$\alpha\varphi^{T}(\Delta - \eta I) = 0 \quad (A9)$$

where α can not be zero, otherwise, there will be zero response. so, we have

$$\varphi^{T}(\Delta - \eta I) = 0 \quad (A10)$$

It is seen from Eqn.(A10) that $\eta = \rho\Delta_{tt}/\lambda$ and φ are the eigen value and eigenvector of the matix Δ respectively. Because φ is the basic vector of the anisotropic subspace of elastic body, if we project the geometric differential operator of dynamical equation on the standard space, its eigenvalue will be proportional to the time differential operator.

From Eqn.(A10), we obtain the following equations:

$$\Delta\varphi_i = \eta\varphi_i \quad (i = 1, 2, \cdots, 6) \quad (A11)$$

$$\Delta\Phi = \Phi\Gamma \quad (A12)$$

$$\Delta = \Phi\Gamma\Phi^{-1} = \Phi\Gamma\Phi^{T} \quad (A13)$$

where Γ is the matrix of eigen geometric differential operator, which is diagonal.

Substituting Eqn.(A13) into Eqn.(A2), we have

$$\Phi^{T}\sigma = \rho\Delta_{tt}\Gamma^{-1}\Phi^{T}e \quad (A14)$$

According to the concept of the elastic standard space:

$$\sigma^{*} = \Phi^{T}\sigma \quad (A15)$$

$$e^{*} = \Phi^{T}e \quad (A16)$$

where σ^{*} and e^{*} are the modal stress vector and modal strain vector respectively.

Eqn.(A14) becomes:

$$\sigma^{*} = \rho\Delta_{tt}\Gamma^{-1}e^{*} \quad (A17)$$

or

$$\Gamma\sigma^{*} = \rho\Delta_{tt}e^{*} \quad (A18)$$

Rewriting Eqn.(A18) in the form of component.

$$\eta_i\sigma_i^{*} = \rho\Delta_{tt}e_i^{*} \quad (i = 1, 2, \cdots, M) \quad (A19)$$

where M is the number of independent anisotropic subspaces.

Using the modal Hooke's Law, Eqn.(A19) become:

$$\eta_i\lambda_i e_i^{*} = \rho\Delta_{tt}e_i^{*} \quad (i = 1, 2, \cdots, M) \quad (A20)$$

This is the eigen form of elastic dynamical equation.

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