GENERALIZED CONJUGATE-GRADIENT ALGORITHM AND ITS APPLICATIONS TO SEISMIC TRACE INVERSION①

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ABSTRACT A novel generalized conjugate-gradient algorithm for complicated equations of seismic trace inverse problems, which is based on classical conjugate-gradient algorithm, has been put forward so as to improve the stability of seismic trace inversion, and to reduce inversion computation and memory resources needed. The algorithm brings high accuracy, fast operation speed and good ability of resisting ill-condition. In addition, by analysing sensitivity matrix according to the specific problem of seismic trace inversion, a new recursive algorithm which needs no sensitivity matrix is developed to save memory greatly. Furthermore, in the new algorithm, sensitivity matrix operation can be converted into convolution and correlation operations to make the whole recursion to be implemented completely by vector operation, which thus speeds recursion operation greatly.

Key words seismic trace inversion conjugate-gradient algorithm accuracy stability operation speed

1 INTRODUCTION

Currently, seismic trace inversion has already been an important work in seismic data processing for meticulous oil-gas exploration, especially for reservoir description and oil-gas lateral prediction. In order to improve the quality of inversion results, i.e., accuracy, stability and resolution, geophysists have put forward different inversion methods in different periods, such as the early generalized linear inversion[1] and recent constrained inversion by using geologic information, seismic data and logs comprehensively[2,3], etc. Certainly, the quality of inversion results depends mainly on the advantages of inversion method itself, but it is also closely related to numerical algorithm for solving the inverse problem. In accordance with ill-posed properties of inverse problem, many mathematical models and their corresponding algorithms have been presented[4-6]. And the well-recognized singular value decomposition (SVD) method, which is characterized by its absolute stability and strong capability to resist ill-condition, should be in the first place. But as well known, SVD algorithm has defects of quite time-consumming and harsh requirement for memory resources. In order to reduce computation and computer memory needed, meanwhile, to enable the numerical algorithm to be applicable to more complicated and more general inversion equations, Zhou[2] has first put forward a generalized conjugate-gradient algorithm, which does almost not lost any precision.

The purpose of this paper is to apply generalized conjugate-gradient algorithm to the specific problems of seismic trace inversion. Inspired by Ref.[1], we have presented some specially generalized conjugate-gradient algorithms for solving the specific problems of seismic trace generalized linear inversion, stochastic inversion and direct impedance inversion. All the algorithms mentioned above need not calculate the coefficient matrix anymore, and the processes

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may be implemented completely by vector operation, which therefore makes us reach the aim of improving accuracy and stability of inversion results, and decreasing computation and saving computer storage greatly.

2 GENERALIZED CONJUGATE-GRADIENT ALGORITHM

Consider the problem of solving linear equations

$$Ax = b \tag{1}$$

where A denotes a coefficient matrix, x a solution vector, and b a data vector. Particularly, for inverse problems, A stands for Jacobi matrix or sensitivity matrix, x usually means model update vector, and b indicates residual vector which is obtained by subtract model response from observed data.

Eqn.(1) is often contradictious for inverse problems, therefore, it is better to regularize it to the following normal equations in sense of least-squares.

$$A^{T}Ax = A^{T}b \tag{2}$$

where A^{T} denotes the transpose of matrix A.

It may be proved that: ① $A^{T}A$ is symmetric and positive semidefinite, hence Eqn.(2) has a unique solution in the sense of L_2 norm; ② the condition number of matrix $A^{T}A$ is the square of that of matrix A, so that the ill-condition of $A^{T}A$ is more serious than that of A, so it is necessary to modify Eqn.(2) to the following damped normal equations

$$(A^{T}A + Id)\,x = A^{T}b \tag{3}$$

where I is an identity matrix, $d = (d_1, d_2, \cdots, d_M)^{T}$ is a vector with damping factors $d_i > 0$ (for $i = 1, 2, \cdots, M$).

Adding the term Id to Eqn.(2) will greatly improve the equation system, and make the procedure of solving more stable. For linear equations shown as Eqn.(3), Zhou[7] has first deduced a generalized conjugate-gradient algorithm which is similar to the standard recursive algorithm of the classical conjugate-gradient method

$$\alpha_j = (g^{(j)}, g^{(j)})/[(Ap^{(j)}, Ap^{(j)}) + (p^{(j)}, q^{(j)})] \tag{4-1}$$

$$x^{(j+1)} = x^{(j)} + \alpha_j p^{(j)} \tag{4-2}$$

$$h^{(j+1)} = h^{(j)} - \alpha_j Ap^{(j)} \tag{4-3}$$

$$g^{(j+1)} = A^{T}h^{(j+1)} - y^{(j+1)} \tag{4-4}$$

$$\beta_{j+1} = (g^{(j+1)}, g^{(j+1)}) / (g^{(j)}, g^{(j)}) \tag{4-5}$$

$$p^{(j+1)} = g^{(j+1)} + \beta_{j+1}p^{(j)} \tag{4-6}$$

where $(\cdot,\cdot)$ represents inner product, j means iteration number in the process of recursion, p and g are gradient and conjugate-gradient vectors separately; $q_i^{(j)} = d_i p_i^{(j)}$, $y_i^{(j)} = d_i x_i^{(j)}$; (for $i = 1, 2, \cdots, M$), and $q_i^{(j)}$, $p_i^{(j)}$, $y_i^{(j)}$, $x_i^{(j)}$ denote the ith element of $q^{(j)}$, $p^{(j)}$, $y^{(j)}$, $x^{(j)}$ respectively; both α_j and β_{j+1} are scalars, which indicate update factors or update steps for x and p respectively. Moreover, when $j = 0$, we have

$$\begin{cases} h^{(0)} = b - Ax^{(0)} \\ g^{(0)} = p^{(0)} = A^{T}h^{(0)} - y^{(0)} \end{cases} \tag{5}$$

where $x^{(0)}$ is the initial estimation of solution vector. Generally, we also call h residual vector, since the more explicit meaning of h is

$$h^{(j)} = b - Ax^{(j)} \tag{6}$$

3 GENERALIZED LINEAR INVERSION OF SEISMIC TRACE WITH GENERALIZED CONJUGATE-GRADIENT METHOD

Let $s(t)$, $s'(t)$, $r(t)$ and $w(t)$ represent actual seismic trace, model response, reflectivity and wavelet separately. Sampled at discrete intervals of Δt, rewritten as s_i and s'_i (for $i = 1, 2, \cdots, N$), r_j (for $j = 1, 2, \cdots, M$), and w_k (for $k = 1, 2, \cdots, L$). Normally, $M \leqslant N$, $L \leqslant M$.

According to Robinson's time-invariant convolution model

$$s'_i = \sum_{j=1}^{N} r_j w_{i-j} \qquad (i = 1, 2, \cdots, N) \tag{7}$$

and least-squares principle, define the misfit function of an inverse problem as

$$\begin{aligned} O(\varepsilon) &= \sum_{i=1}^{N}(s_i - s'_i)^2 \\ &= \sum_{i=1}^{N}\Big(s_i - \sum_{j=1}^{M} r_j w_{i-j}\Big)^2 \\ &= \varepsilon^2 \to \min \end{aligned} \tag{8}$$

then we get

$$\sum_{j=1}^{M} r_j \sum_{i=1}^{N} w_{i-j}w_{i-n} = \sum_{i=1}^{N} s_i w_{i-n}$$

$$(n = 1, 2, \cdots, M) \quad (9)$$

If we let $V = \{V_{i,j}\} = \{w_{i-j}\}$ be wavelet matrix, then Eqn.(9) may be condensed as matrix form

$$V^{\mathrm{T}} V r = V^{\mathrm{T}} s \quad (10)$$

where $r = (r_1, r_2, \cdots, r_M)^{\mathrm{T}}$, $s = (s_1, s_2, \cdots, s_N)^{\mathrm{T}}$ are both column vectors.

If we further let $A = V^{\mathrm{T}} V$, $b = V^{\mathrm{T}} s$, then A is a square, symmetric and positive semidefinite matrix with dimensions $M \times M$, and b is a column vector with dimension M. Therefore, Eqn.(10) may be even simplified as

$$A r = b \quad (11)$$

which is a similar version of the linear Eqn.(1).

Eqn.(10) describes a linear inversion system of seismic trace. In practice, the linear inverse problem of seismic trace is only a special case of seismic trace inverse problems in ideal condition. If the length of actual seismic trace (N) equals the length of reflectivity function (M), then theoretically, the linear inverse problem of seismic trace has an exact solution, hence, Eqn.(10) may be solved directly, no iteration needed. Cardimona[8](1991) has proved that the above linear inverse problem is actually equivalent to deconvolution. But in fact, since real observed data inevitably include noise, moreover, in order to improve the stability of inversion procedure, we usually need add damping term to coefficient matrix to overcome its severe ill-condition, all of these may break down the balance of original exact equations, so the inverse problem itself must be solved in iterative manner in sense of least-squares. For this purpose, we modify the normal equations shown as Eqn.(10) to a new version which fits the generalized linear inverse problem[1] (Cooke *et al*, 1983)

$$(V^{\mathrm{T}} V + I d_n) \Delta r_n = V^{\mathrm{T}} \Delta s_n \quad (12-1)$$

$$r_{n+1} = r_n + \Delta r_n \quad (12-2)$$

$$\Delta s_n = s - s'_n \quad (12-3)$$

where n denotes iteration number within inversion procedure, Δs_n and Δr_n represent data residual vector and reflectivity update vector separately, d_n is a vector conserved damping factors which may be gradually revised versus the change of iteration number n.

Comparing Eqn.(12-1) with Eqn.(3), we may obtain a recursive algorithm for solving the above generalized linear inverse problem of seismic trace based on the generalized conjugate-gradient method

$$\alpha_l = \frac{(g^{(l)}, g^{(l)})}{(Vp^{(l)}, Vp^{(l)}) + (p^{(l)}, q^{(l)})} \quad (13-1)$$

$$\Delta r_n^{(l+1)} = \Delta r_n^{(l)} + \alpha_l p^{(l)} \quad (13-2)$$

$$h^{(l+1)} = h^{(l)} - \alpha_l V p^{(l)} \quad (13-3)$$

$$g^{(l+1)} = V^{\mathrm{T}} h^{(l+1)} - y^{(l+1)} \quad (13-4)$$

$$\beta_{l+1} = (g^{(l+1)}, g^{(l+1)}) / (g^{(l)}, g^{(l)}) \quad (13-5)$$

$$p^{(l+1)} = g^{(l+1)} + \beta_{l+1} p^{(l)} \quad (13-6)$$

where

$$\begin{cases} y^{(l)}(j) = d_n(j) \Delta r_n^{(l)}(j) \\ q^{(l)}(j) = d_n(j) p^{(l)}(j) \end{cases} \quad (j = 1, 2, \cdots, M) \quad (14)$$

and when $j = 0$,

$$\begin{cases} h^{(0)} = \Delta s_n - V \Delta r_n^{(0)} \\ g^{(0)} = p^{(0)} = V^{\mathrm{T}} h^{(0)} - y^{(0)} \end{cases} \quad (15)$$

The principal computation for solving above recursive equations comes from composing of matrix V and calculating of $Vp^{(l)}$, $V\Delta r_n^{(l)}$ and $V^{\mathrm{T}} h^{(l)}$. In order to improve the operation speed of recursion procedure, and save memory needed, we regard $Vp^{(l)}$ and $V\Delta r_n^{(l)}$ and $V^{\mathrm{T}} h^{(l)}$ as vectors, and carry out their elements explicitly for the specific problem of seismic trace generalized linear inversion.

$$Vp^{(l)}(i) = \sum_{j=1}^{M} w_{i-j} p^{(l)}(j) \quad (i = 1, 2, \cdots, N) \quad (16-1)$$

$$V\Delta r_n^{(l)}(i) = \sum_{j=1}^{M} w_{i-j} \Delta r_n^{(l)}(j) \quad (i = 1, 2, \cdots, N) \quad (16-2)$$

$$V^{\mathrm{T}} h^{(l)}(j) = \sum_{i=1}^{N} w_{i-j} h^{(l)}(i) \quad (j = 1, 2, \cdots, M) \quad (16-3)$$

Obviously, it is unnecessary to preserve wavelet matrix V in the procedure of computation, and the calculation of vectors $Vp^{(l)}$, $V\Delta r_n^{(l)}$ and $V^{\mathrm{T}} h^{(l)}$ has been reduced to convolution-type operation described by Eqns.(16-1), (16-2) and correlation-type by Eqn.(16-3). Therefore, the whole recursion procedure has completely been vectorized, which may be handled efficiently by modern array processors.

4 STOCHASTIC INVERSION OF SEISMIC TRACE

To do seismic trace inversion using Eqn. (12), we must know a priori information about damping factors. Formerly, they were figured out by our experience, hence, the inversion results were influenced largely by subjectivity. But it is shown by a vast amount of practice that the damping factors have a great influence on the stability and resolution of inversion results, therefore, Zhou[9] start from the view-point of probability, and on the premise of assumption that both model variance ϕ^2 and data noise variance ψ^2 are random series with null average values, has established a nonlinear system equations as the similar form of Eqn.(12), and given its damping factors the explicit meaning as

$$d_n(j) = \psi^2(j) / \phi_n^2(j) \quad (j = 1, 2, \cdots, M) \qquad (17)$$

where $\phi_n^2(j)$ is elements of model variance vector at the nth iteration in Eqn.(12).

It is thus evident that the distinction between seismic trace stochastic inversion and seismic trace generalized linear inversion lies only in contents of the damping-factor vector. Consequently, the recursive equations described by Eqns.(13) ~ (16) are absolutely applicable to the problem of solving seismic trace stochastic inversion. However, since the damping-factor vector has been endowed with the specific mathematical meaning by expression Eqn. (17), so the stochastic inversion is essentially different from the generalized linear inversion, where the stochastic inversion is a nonlinear inversion in the strict sense.

5 IMPEDANCE INVERSION OF SEISMIC TRACE

The problem of all methods which use the results of reflectivity inversion to reconstruct acoustic impedance of underground media through recursion formula is that, the direct component of reflectivity is often hard to control, and because of accumulation of errors in the recursion procedure, a small error of the direct component of reflectivity may result in a serious shift of the reconstructed acoustic impedance curve. To solve this problem, the model-based inversion method which recover acoustic impedance directly is developed in recent years.

Zhou[9] had revealed the impedance inversion of seismic trace which has the following form of inversion system equations in a similar way to Eqn.(12)

$$(\boldsymbol{V}^{\mathrm{T}}\boldsymbol{V} + \boldsymbol{I}\boldsymbol{d}_n)\Delta \boldsymbol{z}_n = \boldsymbol{V}^{\mathrm{T}}\Delta \boldsymbol{z}_n \qquad (18-1)$$

$$\boldsymbol{z}_{n+1} = \boldsymbol{z}_n + \Delta \boldsymbol{z}_n \qquad (18-2)$$

$$\Delta \boldsymbol{s}_n = \boldsymbol{s} - \boldsymbol{s}'_n \qquad (18-3)$$

where $\boldsymbol{z}_n$, $\boldsymbol{z}_{n+1}$ are impedance vectors to be recovered at two neighbouring iterations, $\Delta \boldsymbol{z}_n$ represents a vector of impedance updates.

In this case, the elements of coefficient matrix $\boldsymbol{V}$ should be correspondingly modified to

$$V_{i,j} = \xi_n(j)\, w_{i+1-j} - \eta_n(j)\, w_{i-j} \qquad (19)$$

where

$$\begin{cases} \xi_n(j) = 2z_n(j-1) / \\ \quad [z_n(j) + z_n(j-1)]^2 \\ \eta_n(j) = 2z_n(j+1) / \\ \quad [z_n(j) + z_n(j+1)]^2 \end{cases} \qquad (20)$$

We may use the recursive algorithm of generalized conjugate-gradient method which is similar to Eqns.(13) ~ (16) to solve this inverse problem also, but the expressions for calculating the coefficient matrix related vectors $\boldsymbol{V}\boldsymbol{p}^{(l)}$, $\boldsymbol{V}\Delta \boldsymbol{z}_n^{(l)}$ and $\boldsymbol{V}^{\mathrm{T}}\boldsymbol{h}^{(l)}$ must be modified.

6 RESULTS FROM FIELD DATA TEST

The purpose of test: ① to verify the validity of our inversion algorithm obtained by applying generalized conjugate-gradient (GCG) algorithm to the specific inverse problem of seismic trace (We simply call it "specific algorithm" in following text); ② to compare some technical indices (such as accuracy, operation time and memory needed, etc.) of SVD algorithm, GCG algorithm and the specific algorithm. The field data we used, shown in Fig.1, is a windowed seismic time section corresponding to CDP 1 675

~1 725 and time 1 250 ~1 650 ms on a line from North China area, and is obtained by using of amplitude preserved processing. There is a well located at CDP 1698 which may be used to check the results of seismic inversion. After inversion with the GCG algorithm, we get the final model response shown in Fig.2. Comparing Fig.1 with Fig.2, we may easily find that they are in good match. Fig.3 illustrates the final impedance section as a result of inversion, which may help geologists even better to delineate seismic and geological interfaces, and to determine lateral variations of reservoir properties. The bold curve inserted in this map shows the actual impedance from the well-logs. Comparing the results of impedance inversion with well-logs, we may obviously see that the results of inversion coincide with actual impedance very well, excepting two misfits at segments 1520 ~ 1540 ms and 1560 ~ 1580 ms possibly due to multiples. The good effect of this example indicates that the inversion algorithm we deduced is completely valid.

In addition, we have also used the previous SVD and GCG algorithms to solve the same inverse problem as above example numerically. The results obtained are fully identical with Fig.1 to Fig.3 correspondingly. It is invisible to identify their differences with naked eyes at all. But the two technical indices, operation time and memory space needed, are quite different. Table.1 gives us a direct perception on some technical indices of the three recursive algorithms used for the example. According to Table.1, when replaced SVD algorithm by GCG algori-

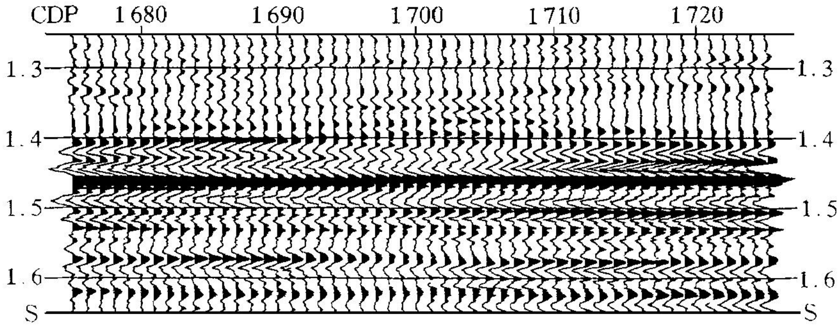

Fig.1 A windowed seismic time section on a line in North China area, there is a well located at CDP1698

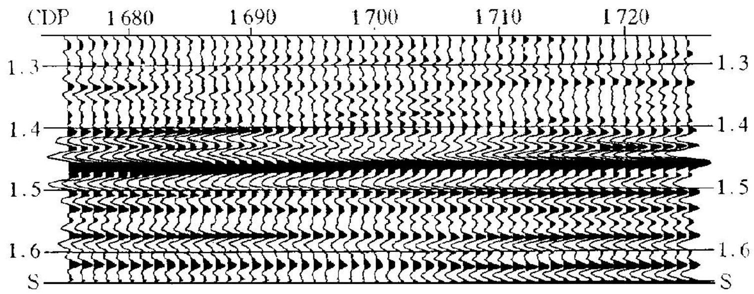

Fig.2 Final model response of inversion

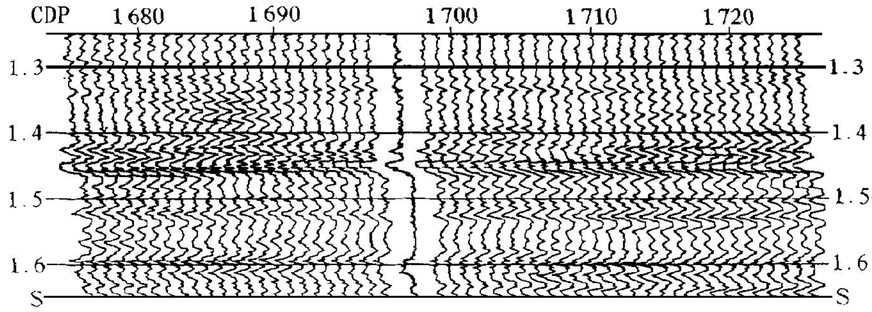

Fig.3 Final output impedance section of inversion, the bold curve inserted shows the actual impedance from well-logs

Table 1 Comparisons of technical indices of three inversion algorithms

Technical indices	SVD algorithm	GCG algorithm	Specific algorithm
Accuracy	0.9837	0.9818	0.9818
Operation time	85.6	4.3	1.0
Memory space (unit)	41107	10908	909

For the case: trace length(N) is 101 samples and wavelet length (L) 21 samples, where accuracy is measured by correlation between inversion results and well-logs, operation time is related to that of specific algorithm spent.

thm, the operation time and memory space needed have been decreased significantly, especially the operation time has been reduced essentially, while the inversion accuracy has almost not been affected. Furthermore, when applied GCG algorithm to the specific problem of seismic inversion, the accuracy has not been changed any more, and the operation speed has been improved in a certain degree, but the memory requirement has been reduced greatly. Table 2 demonstrates the approximate estimations of the operation time and the memory requirement for the three different inversion algorithms. As can be seen, after replacing SVD algorithm by the specific algorithm deduced by this paper, both the operation time and the memory requirement are reduced in one order about N. This indicates that, with increasing of the length (N) of seismic trace to be inversed, the specific algorithm is incomparably superior to SVD algorithm.

Table 2 Approximate estimations of operation time and memory requirement for three different inversion algorithms

Technical indices	SVD algorithm	GCG algorithm	Specific algorithm
Operation time*	$O(N^3)$	$O(N^2)$	$O(N^2)\cdot L/N$
Memory space (unit)	$4N^2+3N$	N^2+7N	$9N$

* where $O(\cdot)$ represents "order"

7 CONCLUSIONS

By applying generalized conjugate-gradient algorithm to inverse problem of seismic trace, we have obtained the recursive algorithms for different seismic trace inversion methods. As compared with the previous well-recognized SVD algorithm, new algorithms possess following advantages:

(1) Because of applying generalized conjugate-gradient algorithm, the calculation amount of inversion has been dropped from order $O(N^3)$ to $O(N^2)$, that is, has been reduced in one order. Furthermore, because of decomposing matrix operation into vector convolution and correlation, the calculation amount has been even more reduced. This is significant to the present

large-scaled seismic inversion.

(2) Since the coefficent matrix needs not to be saved, the requirement of computer memory has been dropped one order. This makes the requirement for computer resources no longer harsh, and makes the implementation of multi-channel inversion technique to be more convenient.

(3) The accuracy of inversion results is hardly ever influenced by algorithm itself.

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