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Operationalized principle of elastic mechanics①

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[Abstract] The fundamental mechanical equations were studied under the mechanical space. The differential stress operator and strain operator were obtained. There were strain energy operator and Hamilton operator for elastic body in same way, and the following results were testified. 1) The equilibrium equation of force is equivalent to the harmony equation of deformation under the mechanical space. They are all the basic mode of eigen equation of stress or strain operator. 2) The eigen value of stress or strain operator is corresponding to the order of kinetic energy of elastic body, and the elastic wave represents the non-basic mode. 3) The eigen functions of stress operator or strain operator corresponding to some kinetic energy order are fields of modal stress or modal strain in same order. 4) The eigen equations of strain energy operator are the fundamental equations of elastic mechanics which are expressed with the potential functions.

[Key words] anisotropic solid; elastic mechanics; mechanical operator; operationalized principle

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1 INTRODUCTION

The dividing phenomenon of motion, which is described in quantum mechanics, is the general law in material world. It exists not only in micromechanism, but also in continuum. For example, when studying the mechanical vibration and the elastic wave of solid, when studying the motion of fluid and the propagation of voice, we can see the dividing mechanical quantities, such as frequency, amplitude and simple harmonic motion. As to static mechanics, it is only the basic mode of the dividing motion. For the classical elastic mechanics, the elastic body is not studied in the view of the dividing motion with the operationalized principle, which is compatible with quantum mechanics, but by using the geometrical principle of Newton's law, the equilibrium equation of force and the harmony equation of deformation in the form of tensor are given, and many branching theories are formed, such as elastic mechanics, elastic dynamics, isotropic mechanics and anisotropic mechanics. The inadequacy of the classical elastic mechanics is that it can not describe the varied anisotropic elastic body in a unitized and explicit form, and can not give the regularized equations for the different mechanical processes.

In order to change the system of classical elastic mechanics into one of quantum mechanics, it is necessary to alter some traditional ideas. For example, replacing the geometrical space with the mechanical space, the mechanical quantities with the mechanical operators, and the differential equations with the operator equations. To overcome the theoretical difficulty of classical elastic mechanics, elastic mechanics are improved with the operationalized principle[1~7].

2 DEFINITION OF MECHANICAL SPACE

The matrix form of the generalized Hooke's law under geometrical space is

$$\sigma = C\varepsilon \tag{1}$$

Eqn.(1) holds the eigen equation of elasticity[8~10].

$$(C - \lambda I)\varphi = 0 \tag{2}$$

where λ and φ are eigen value and eigen vector of elastic coefficients matrix C, respectively. λ is Kelvin elastic module, and is not related to coordinates. φ is mechanical space, and indicates the anisotropic direction of elastic body. Thus the elastic coefficients matrix under geometrical space can be decomposed spectrally under mechanical space

$$C = \Phi\Lambda\Phi^{T} \tag{3}$$

where Φ is eigen modal matrix of elastic body, and is orthogonal and symmetric. Λ is eigen elastic matrix and is diagonal.

So, the generalized Hooke's law under mechanical space becomes the normal form

$$\sigma_i^* = \lambda_i \varepsilon_i^*, \qquad i = 1, 2, \cdots, 6 \tag{4}$$

Eqn.(4) is called the modal Hooke's law in which the modal stress and strain vector are

$$\sigma^* = \Phi^{T}\sigma \tag{5}$$

$$\varepsilon^* = \Phi^{T}\varepsilon \tag{6}$$

3 STRESS AND STRAIN OPERATORS

Under the geometrical space, the equilibrium equation of force and the harmony equation of deformation of elastic body respectively are[6,7]

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$$\sigma_{ik,kj} + \sigma_{jk,ki} = 0 \tag{7}$$

$$e_{ij,kl} - e_{kj,il} + e_{kl,ij} - e_{il,kj} = 0 \tag{8}$$

Using the symmetry of index, Eqns. (7) and (8) can be written as the form of matrix

$$[\Delta]\sigma = 0 \tag{9}$$

$$[\nabla]\varepsilon = 0 \tag{10}$$

where $[\Delta]$ and $[\nabla]$ are symmetrical differential operator matrices of the second order and called stress operator matrix and strain operator matrix respectively.

$$[\Delta] = \begin{bmatrix} \partial_{11} & 0 & 0 & 0 & \partial_{31} & \partial_{21} \\ & \partial_{22} & 0 & \partial_{32} & 0 & \partial_{21} \\ & & \partial_{33} & \partial_{32} & \partial_{31} & 0 \\ & & & (\partial_{22}+\partial_{33}) & \partial_{21} & \partial_{31} \\ & & & & (\partial_{11}+\partial_{33}) & \partial_{32} \\ & & & & & (\partial_{22}+\partial_{11}) \end{bmatrix} \tag{11}$$

$$[\nabla] = \begin{bmatrix} 0 & \partial_{33} & \partial_{22} & -\partial_{23} & 0 & 0 \\ & 0 & \partial_{11} & 0 & -\partial_{13} & 0 \\ & & 0 & 0 & 0 & -\partial_{12} \\ & & & -\frac{1}{2}\partial_{11} & \frac{1}{2}\partial_{12} & \frac{1}{2}\partial_{13} \\ & & & & -\frac{1}{2}\partial_{22} & \frac{1}{2}\partial_{23} \\ & & & & & -\frac{1}{2}\partial_{33} \end{bmatrix} \tag{12}$$

where $\partial_{ij} = \partial_{ji} = \partial^2 / \partial x_i \partial x_j$

Projecting Eqns. (9) and (10) into the mechanical space, the following equations are proved[6,7]

$$[\Delta] = \Phi^{\mathrm{T}}[\Delta^*]\Phi \tag{13}$$

$$[\nabla] = \Phi^{\mathrm{T}}[\nabla^*]\Phi \tag{14}$$

where $[\Delta^*]$ and $[\nabla^*]$ are diagonal and called eigen stress and strain differential operator matrix respectively. Thus, under mechanical space, the equilibrium equation of force and the harmony equation of deformation of elastic body become[6,7]

$$\Delta_i^* \sigma_i^* = 0, \quad i = 1, 2, \cdots, 6 \tag{15}$$

$$\nabla_i^* \varepsilon_i^* = 0, \quad i = 1, 2, \cdots, 6 \tag{16}$$

where Δ_i^* and ∇_i^* are called stress operator and strain operator respectively.

Using Eqn. (4), and comparing Eqn. (15) with Eqn. (16), We get

$$\nabla_i^* = \lambda_i \Delta_i^*, \quad i = 1, 2, \cdots, 6 \tag{17}$$

Thus under mechanical space, the equilibrium equation of force is equivalent to the harmony equation of deformation in the process of physics.

4 HAMILTON OPERATOR OF ELASTIC BODY

Under the field of potential force, Hamilton quantity of elastic body is sum of the kinetic energy and potential energy. If replacing the mechanical quantity with the mechanical operator, the Hamilton operator is obtained

$$\hat{H} = \hat{T} + V(r) \tag{18}$$

where $\hat{T}$ is the kinetic energy operator, and $V(r)$ is the potential energy.

Because the operator of velocity vector is differentiation of the first order with respect to time, the kinetic energy operator is differentiation of the second order with respect to time.

$$\hat{T} = \frac{1}{2}\rho \nabla_{tt} \tag{19}$$

where ρ is the intensity of elastic body, and $\nabla_{tt} = \partial^2 / \partial t \partial t$.

It is proved[6] that the geometrical differential operator of stress or strain is in direct proportion to that of time for elastic body

$$\nabla_{tt} = \frac{\lambda_i \Delta_i^*}{\rho}, \quad i = 1, 2, \cdots, 6 \tag{20}$$

Thus, when neglecting the potential force, the component of the Hamilton operator under mechanical space becomes

$$\hat{H}_i = \frac{1}{2}\lambda_i \Delta_i^* \quad i = 1, 2, \cdots, 6 \tag{21}$$

From Eqn. (21), it can be seen that Hamilton operator of elastic body is in direct proportion to the stress operator or strain operator, which gives the meaning of kinetic energy to the deformation of elastic body.

5 EIGEN DIFFERENTIAL EQUATION OF OPERATOR

According to the acting principle of differential operator, when Hamilton operator acts on some function, it is equal to eigen value of the differential operator multiplies by the same function.

$$\hat{H}_i f_i(x) = D_i f_i(x), \quad i = 1, 2, \cdots, 6 \tag{22}$$

Substituting Eqn. (21) into Eqn. (22), then

$$\lambda_i \Delta_i^* f_i(x) = d_i f_i(x), \quad i = 1, 2, \cdots, 6 \tag{23}$$

Considering the physical meaning of Eqn. (23), under the mechanical space, the wave equation of elastic body is given as follows[6]:

$$\lambda_i \Delta_i^* \varepsilon_i^*(x, t) = \rho \nabla_{tt} \varepsilon_i^*(x, t), \quad i = 1, 2, \cdots, 6 \tag{24}$$

Dividing the variables of strain field into two parts,

$$\varepsilon_i^*(x, t) = f_i(x) \cdot g_i(t), \quad i = 1, 2, \cdots, 6 \tag{25}$$

and substituting Eqn. (25) into Eqn. (24), Eqns. (26) and (27) are obtained:

$$\lambda_i \Delta_i^* f_i(x) = d_i f_i(x), \quad i = 1, 2, \cdots, 6 \tag{26}$$

$$\rho \nabla_{tt} g_i(t) = d_i g_i(t), \quad i = 1, 2, \cdots, 6 \tag{27}$$

The eigen differential equation of stress operator is just the static portion of the dynamical equation of elastic body. The eigen value of stress operator means the kinetic energy. When elastic body is in the basic mode, that is, $d_0 = 0$, the eigen differential equation is just the equilibrium equation of force or harmony equation of deformation of the static mechanics of elastic body.

6 STRAIN ENERGY OPERATOR AND ITS EIGEN DIFFERENTIAL EQUATION

Although the eigen differential equation of Hamilton operator has clear physical meaning, it is necessary to study the operator equation in the view of strain energy.

According to the calculating formula of strain energy, replacing stress with stress operator and strain with strain operator, we can define strain energy operator in same way.

$$2U = \sigma \cdot \varepsilon = \sigma^* \cdot \varepsilon^* \quad (28)$$

$$2\hat{U} = [\Delta^*][\nabla^*] = \sum \Delta_i^* \nabla_i^* \quad (29)$$

Using Eqn.(1),

$$\hat{U} = \frac{1}{2}\sum \lambda_i \Delta_i^{*2} = \sum \hat{H}_i \Delta_i^* = \sum \hat{U}_i^* \quad (30)$$

According to the acting principle of the differential operator, the eigen differential equation of strain energy operator is

$$\hat{U}_i^* \phi_i = E_i \phi_i, \quad i = 1, 2, \cdots, 6 \quad (31)$$

Comparing Eqn.(21) and Eqn.(30), it is seen that strain energy operator is in direct proportion to the square of Hamilton operator. So the physical meaning of the eigenvalue of strain energy operator, E_i, is the square of the kinetic energy of elastic body in the corresponding mechanical space.

According to the definition of potential function under mechanical space[7],

$$\varepsilon_i^* = \Delta_i^* \Psi_i, \quad i = 1, 2, \cdots, 6 \quad (32)$$

where Ψ_i is the potential function. Substituting Eqn.(32) into Eqn.(24), and using Eqn.(20), get

$$\lambda_i^2 \Delta_i^{*2} \Psi_i = \rho^2 \nabla_{tttt} \Psi_i, \quad i = 1, 2, \cdots, 6 \quad (33)$$

It is just the wave equation expressed with potential function of elastic body.

Dividing the variable of the potential function into two parts,

$$\Psi_i(x, t) = \phi_i(x) \cdot \gamma_i(t),$$
$$i = 1, 2, \cdots, 6 \quad (34)$$

and substituting Eqn.(34) into Eqn.(33), then

$$\hat{U}_i^* \phi_i(x) = E_i \phi_i(x), \quad i = 1, 2, \cdots, 6 \quad (35)$$

$$\rho^2 \nabla_{tttt} \gamma_i(t) = \lambda_i E_i \gamma_i(t),$$
$$i = 1, 2, \cdots, 6 \quad (36)$$

From Eqn.(35), It is seen that the eigen differential equation of strain energy operator is just the static portion of dynamical equation expressed with the potential function of elastic body. When elastic body is in the basic mode, that is, $E_0 = 0$, it is obtained from Eqn.(35) that

$$\Delta_i^* \nabla_i^* \phi_0(x) = 0, \quad i = 1, 2, \cdots, 6 \quad (37)$$

Eqn.(37) is just the double harmony equation of elastic mechanics, and ϕ_0 is just the stress function. Thus, there exist six independent stress functions for the general anisotropic body. The definite equation and its boundary condition for the varied anisotropic body was given in Ref.[7].

7 CONCLUSIONS

The elastic mechanics under mechanical space is studied, and the operationalized principle of elastic mechanics is presented. Compared with the classical elastic mechanics under geometrical space, the new elastic mechanics exhibits: 1) the modal stress (modal strain) becomes the measure of mechanical condition; 2) the eigen equation of stress operator (strain operator) becomes the fundamental equation of elastic mechanics; and 3) the order of kinetic energy composes the spectral distribution of motion of elastic body.

The advantages of the operationalized principle of elastic body are 1) no matter how complicated the anisotropy of elastic body may be, there exist a set of definite and explicit formulae; 2) there is no difference between force method and displacement method; 3) there is no methodology difference between static mechanics and dynamics; 4) the cutting technique of mode can be used to simplify the calculation of elastic anisotropic body.

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