Article ID: 1003- 6326(2000)01- 0123- 04

Estimation and accuracy evaluation of variance and covariance components based on more general functional model ①

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Abstract: Started from the more general functional model and based on the work of Koch K R (1986) and QU Zi-qiang (1989), marginal likelihood function of variance and covariance components is derived and is identical with the orthogonal complement likelihood function. Minimum norm quadratic unibiased estimator (MINQUE) is developed, which expands the formula by Rao C R (1973). It is proved that Helmert type estimation, MINQUE, BQUE(Best quadratic unibiased estimation) and maximum likelihood estimation are identical with one another. Besides, a universal formula for accuracy evalution is presented. Through these work, a universal theory of variance and covariance components is established.

Key words: variance component; marginal likelihood; accuracy evaluation **Document code:** A

1 INTRODUCTION

The study of variance estimation is one of the most important fields on geodesy and mathematical statistics. The variance component estimation based on the likelihood principle and minimum norm principle has been derived, using the functional models of condition adjustment with unknown parameter and parameter adjustment[1,2]. However, the unknown parameters in the models are functionally independent, therefore they cannot be applied to conditions in which dependent parameters are involved. To improve this case, Yu[3] has given a universal formula of maximum likelihood estimation of variance and covariance components under the general functional model. The aim of this paper is trying to establish a universal theory of estimation and accuracy evaluation of variance and covariance components.

2 MARGINAL LIKELIHOOD AND ORTHOGONAL COMPLEMENT LIKELIHOOD

Considering the more general functional model

$$\underset{c,nn,1}{\boldsymbol{A}\boldsymbol{V}} + \underset{c,uu,1}{\boldsymbol{B}\hat{\boldsymbol{x}}} - \underset{c,1}{\boldsymbol{f}} = 0 \tag{1a}$$

$$\underset{s,uu,1}{\boldsymbol{C}\hat{\boldsymbol{x}}} - \underset{s,1}{\boldsymbol{f}_x} = 0 \tag{1b}$$

$$R(\boldsymbol{A}) = c,\ R(\boldsymbol{B}) = u,\ R(\boldsymbol{C}) = s \tag{1c}$$

where $-\boldsymbol{f} = \boldsymbol{AL} + \boldsymbol{BX}^\circ + \boldsymbol{A}^\circ$; $-\boldsymbol{f}_x^\circ = \boldsymbol{CX}^\circ + \boldsymbol{C}^\circ$; $\underset{n,1}{\boldsymbol{L}}$ is the observation vector; $\boldsymbol{X}^\circ$, $\boldsymbol{A}^\circ$ and $\boldsymbol{B}^\circ$ are the vectors of approximative values of parameters and known coefficient matrices, respectively; $\boldsymbol{V}$ and $\hat{\boldsymbol{x}}$ are the correction vectors of $\boldsymbol{L}$ and $\boldsymbol{X}^\circ$, respectively; $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are known coefficient matrices. Assume

$$\boldsymbol{L} \sim N(\mu_L,\ \boldsymbol{D}) \tag{2}$$

where μ_L is expectation of $\boldsymbol{L}$; $\boldsymbol{D}$ is covariance matrix of $\boldsymbol{L}$; $\boldsymbol{D} = \sigma_0^2\boldsymbol{Q}$, $\boldsymbol{Q}$ is cofactor matrix of $\boldsymbol{L}$.

Suppose that $\boldsymbol{B}$ and $\boldsymbol{C}$ from (1) are

$$\left.\begin{array}{l} \boldsymbol{B} = [\boldsymbol{B}_1,\ \boldsymbol{B}_2],\ \boldsymbol{C} = [\boldsymbol{C}_1,\ \boldsymbol{C}_2] \\ \hat{\boldsymbol{x}} = [\hat{\boldsymbol{x}}_1^{\mathrm{T}},\ \hat{\boldsymbol{x}}_2^{\mathrm{T}}]^{\mathrm{T}},\ u = u_1 + s \end{array}\right\} \tag{3}$$

and $\boldsymbol{C}_2$ is an invertible matrix of order s, then Eqn. (1) can be rewritten as

$$\underset{c,nn,1}{\boldsymbol{A}\boldsymbol{V}} + \underset{c,u\,u_1,1}{\bar{\boldsymbol{B}}\hat{\boldsymbol{x}}_1} - \underset{c,1}{\bar{\boldsymbol{f}}} = 0 \tag{4}$$

where $\bar{\boldsymbol{B}} = \boldsymbol{B}_1 - \boldsymbol{B}_2\boldsymbol{C}_2^{-1}\boldsymbol{C}_1$, $\bar{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{B}_2\boldsymbol{C}_2^{-1}\boldsymbol{f}_x$.

If the observations are considered to be divided into m groups, denoted as $\boldsymbol{L}_1$, $\boldsymbol{L}_2$, …, $\boldsymbol{L}_m$, according to heterogeneous types and/or different precisions, Eqn. (4a) can be written as

$$\boldsymbol{A}_1\boldsymbol{V}_1 + \boldsymbol{A}_2\boldsymbol{V}_2 + \cdots + \boldsymbol{A}_m\boldsymbol{V}_m + \bar{\boldsymbol{B}}\hat{\boldsymbol{x}}_1 - \bar{\boldsymbol{f}} = 0 \tag{5}$$

where $\boldsymbol{A} = [\boldsymbol{A}_1,\ \boldsymbol{A}_2,\ \cdots,\ \boldsymbol{A}_m]$, $\boldsymbol{V} = [\boldsymbol{V}_1^{\mathrm{T}},\ \boldsymbol{V}_2^{\mathrm{T}},\ \cdots,\ \boldsymbol{V}_m^{\mathrm{T}}]^{\mathrm{T}}$.

If $\boldsymbol{L}_i$ and $\boldsymbol{L}_j$ are correlated $(i,\ j = 1,\ 2,\ \cdots,\ m)$, and improper initial unit weights are assigned to every subvector of observations, then, the variance matrix of observation vector $\boldsymbol{L}$ should be

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{Q}_{11}\sigma_{01}^2 & \boldsymbol{Q}_{12}\sigma_{012}^2 & \cdots & \boldsymbol{Q}_{1m}\sigma_{01m}^2 \\ \boldsymbol{Q}_{21}\sigma_{012}^2 & \boldsymbol{Q}_{22}\sigma_{02}^2 & \cdots & \boldsymbol{Q}_{2m}\sigma_{2m}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{Q}_{m1}\sigma_{01m}^2 & \boldsymbol{Q}_{m2}\sigma_{02m}^2 & \cdots & \boldsymbol{Q}_{mm}\sigma_{0m}^2 \end{bmatrix} = \sum_{i=1}^{q}\tilde{\boldsymbol{Q}}_i\sigma_i^2 \tag{6}$$

where $q = m(m+1)/2$, the definition of the notations $\tilde{\boldsymbol{Q}}_i$ and σ_i^2 $(i = 1,\ 2,\ \cdots,\ q)$ can be found in [3], from which we have the probability density function of $\bar{\boldsymbol{f}}$:

$$L(\bar{\boldsymbol{f}},\ \boldsymbol{x}_1,\ \sigma_i^2) = \{1/[(2\pi)^{\frac{c}{2}}\det(\dot{\boldsymbol{N}}_{aa})^{1/2}]\} \cdot \exp\{-(\bar{\boldsymbol{f}} - \bar{\boldsymbol{B}}\boldsymbol{x}_1)^{\mathrm{T}}/2 \cdot \dot{\boldsymbol{N}}_{aa}^{-1}(\bar{\boldsymbol{f}} - \bar{\boldsymbol{B}}\boldsymbol{x}_1)\} \tag{7}$$

① **Foundation item:** Project 49774209 supported by the National Natural Science Foundation of China
Received date: Oct. 17, 1998; **accepted date:** Aug. 26, 1999

where $\dot{N}_{aa} = \sum_{i=1}^{q} N_{ai}\sigma_i^2$, $N_{ai} = A\tilde{Q}_i A^{\mathrm{T}}$.

The usual approach is equivalent to let the function $L(\pounds,\ x_1,\ \sigma_i^2)$ to be maximum, so we can solve $\hat{x}_1$ and $\hat{\sigma}_i^2$. The maximum likelihood estimates of σ_i^2 are biased and its results are different from those of any other methods, because we have not taken care of the loss in degrees of freedom from the estimates of the unknown parameters x_1. In order to solve the problem, we firstly obtain the maximum likelihood estimates of the unknown parameters x_1 from (7):

$$\hat{x}_1 = \dot{N}_{bb}^{-1} B^{\mathrm{T}} \dot{N}_{aa}^{-1} \pounds \tag{8}$$

where $\dot{N}_{bb} = B^{\mathrm{T}} \dot{N}_{aa}^{-1} B$.

As

$$(Bx_1 - B\hat{x}_1)^{\mathrm{T}} \dot{N}_{aa}^{-1} (\pounds - B\hat{x}_1) = 0 \tag{9}$$

we have

$$\begin{aligned}(\pounds - Bx_1)^{\mathrm{T}} \dot{N}_{aa}^{-1} (\pounds - Bx_1) = \\ [(\pounds - B\hat{x}_1)^{\mathrm{T}} \dot{N}_{aa}^{-1} (\pounds - B\hat{x}_1) + \\ (x_1 - \hat{x}_1)^{\mathrm{T}} \dot{N}_{bb} (x_1 - \hat{x}_1)\end{aligned} \tag{10}$$

substituting (10) into (7) leads to

$$\begin{aligned}L(\pounds,\ x_1,\ \sigma_i^2) = 1/\left\{(2\pi)^{\frac{c}{2}} \det(\dot{N}_{aa})^{\frac{1}{2}}\right\} \cdot \\ \exp\left\{-\frac{1}{2}(\pounds - B\hat{x}_1)^{\mathrm{T}} \dot{N}_{aa}^{-1} (\pounds - B\hat{x}_1)\right\} \cdot \\ \exp\left\{-\frac{1}{2}(x_1 - \hat{x}_1)^{\mathrm{T}} \dot{N}_{bb}^{-1} (x_1 - \hat{x}_1)\right\}\end{aligned} \tag{11}$$

Integrating Eqn. (11) about x_1, we have

$$\begin{aligned}L(\pounds,\ \sigma_i^2) &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} L(\pounds,\ x_1,\ \sigma_i^2)\,\mathrm{d}x_1 \\ &= \frac{1}{(2\pi)^{\frac{c-u}{2}} [\det \dot{N}_{aa} \det \dot{N}_{bb}]^{\frac{1}{2}}} \cdot \\ &\quad \exp\left\{-\frac{1}{2}(\pounds^{\mathrm{T}} M \pounds)\right\}\end{aligned} \tag{12}$$

where

$$M = \dot{N}_{aa}^{-1} - \dot{N}_{aa}^{-1} B \dot{N}_{bb}^{-1} B^{\mathrm{T}} \dot{N}_{aa}^{-1} \tag{13}$$

Eqn. (12) applies the following formula

$$\begin{aligned}\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\{-\frac{1}{2}(x_1 - \hat{x}_1)^{\mathrm{T}} \cdot \\ \dot{N}_{bb}(x_1 - \hat{x}_1)\}\,\mathrm{d}x_1 \\ = (2\pi)^{\frac{u_1}{2}} \det \dot{N}_{bb}^{-\frac{1}{2}}\end{aligned} \tag{14}$$

The likelihood function $L(\pounds,\ \sigma_i^2)$ does not depend on the unknown parameters x_1, but on the parameter $\sigma_i^2 (i = 1,\ 2,\ \cdots,\ q)$ only. Therefore $L(\pounds,\ \sigma_i^2)$ is called the marginal likelihood function.

From [3], we have the orthogonal complement likelihood function

$$\begin{aligned}L_1(\pounds,\ \sigma_i^2) = \frac{1}{(2\pi)^{\frac{c-u}{2}} \det(\alpha \dot{N}_{aa} \alpha)^{\frac{1}{2}}} \cdot \\ \exp\left\{-\frac{1}{2}\pounds^{\mathrm{T}} \alpha^{\mathrm{T}} (\alpha \dot{N}_{aa} \alpha^{\mathrm{T}})^{-1} \alpha \pounds\right\}\end{aligned} \tag{15}$$

Now, we prove that Eqns. (12) and (15) are identical to each other, choosing α in (15) satisfies

$$\alpha^{\mathrm{T}} \alpha = I - B(B^{\mathrm{T}} B)^{-1} B^{\mathrm{T}} \tag{16}$$

$$G = \begin{vmatrix} \alpha \\ B^{\mathrm{T}} \dot{N}_{aa}^{-1} \end{vmatrix} \tag{17}$$

It is evident that $\alpha B = 0$, $R(\alpha) = c - u_1$, and that maximum likelihood estimation based on $L_1(\pounds,\ \sigma_i^2)$ is invariant, although α may have different choices. These can be proved similarly to [4]. From [3], we get

$$\det GG^{\mathrm{T}} = \det(\dot{N}_{bb})^2 \det(B^{\mathrm{T}} B)^{-1} \tag{18}$$

we can still obtain

$$\begin{aligned}\det \dot{N}_{aa} \det(GG^{\mathrm{T}}) &= \det(G\dot{N}_{aa}G^{\mathrm{T}}) \\ &= \det(\alpha \dot{N}_{aa} \alpha^{\mathrm{T}}) \det(\dot{N}_{bb})\end{aligned} \tag{19}$$

From (18) and (19), we have

$$\det \dot{N}_{aa} \det \dot{N}_{bb} \det(B^{\mathrm{T}} B)^{-1} = \det(\alpha \dot{N}_{aa} \alpha^{\mathrm{T}}) \tag{20}$$

From appendix A in [5], we have

$$\begin{aligned}\alpha^{\mathrm{T}}(\alpha \dot{N}_{aa} \alpha^{\mathrm{T}})\alpha &= \dot{N}_{aa}^{-1} - \dot{N}_{bb}^{-1} B \dot{N}_{bb}^{-1} B^{\mathrm{T}} \dot{N}_{aa}^{-1} \\ &= M\end{aligned} \tag{21}$$

Substituting (20) and (21) into (15) leads to

$$\begin{aligned}L_1(\pounds,\ \sigma_i^2) = \\ \frac{1}{(2\pi)^{\frac{c-u_1}{2}} [\det \dot{N}_{aa} \det \dot{N}_{bb} \det(B^{\mathrm{T}} B)^{-1}]^{\frac{1}{2}}} \cdot \\ \exp\left\{-\frac{1}{2}\pounds^{\mathrm{T}} M \pounds\right\}\end{aligned} \tag{22}$$

Comparing (22) and (12), we get that the marginal likelihood function is identical with the orthogonal complement likelihood function.

3 MINIMUM NORM QUADRATIC UNIBIASED ESTIMATOR (MINQUE)

Considering the model

$$E(\pounds) = Bx_1 \tag{23a}$$

$$D(\pounds) = \sum_{j=1}^{q} \sigma_j^2 N_{aj} = \sum_{j=1}^{q} \sigma_j^2 H_j H_j^{\mathrm{T}} \tag{23b}$$

where $H_i (i = 1,\ 2,\ \cdots,\ q)$ are complex matrices. Because N_{aj} is a symmetrical matrix, there must exist a complex matrix H_j satisfying

$$\underset{n_j,\ n_j}{N_{aj}} = H_j H_j^{\mathrm{T}} \quad (j = 1,\ 2,\ \cdots,\ q) \tag{23c}$$

Assume that random vectors $\xi_1,\ \xi_2,\ \cdots,\ \xi_l$ are independent and normally distributed, i. e.

$$\xi_i \sim N(0,\ \sigma_i^2 I_{ni}) \tag{24}$$

Eqn. (23) can be written as

$$H_1 \xi_1 + H_2 \xi_2 + \cdots + H_q \xi_l + Bx_1 - \pounds = 0 \tag{25}$$

$\pounds$ in (5a) and (25) has the same expectation and the covariance matrices and is normally distributed, therefore $\pounds$ in (5a) and in (25) is identified

Now, let's estimate the linear function

$$\varphi = \alpha_1 \sigma_1^2 + \alpha_2 \sigma_2^2 + \cdots \alpha_k \sigma_q^2 = \alpha^{\mathrm{T}} \theta \tag{26}$$

where

$$\theta = [\sigma_1^2,\ \sigma_2^2,\ \cdots,\ \sigma_q^2]^T \tag{27}$$

$$\alpha = [\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_q]^T \tag{28}$$

Naturally, we choose the quadratic form of observation function

$$\varphi = \boldsymbol{\pounds}^T \Omega \boldsymbol{\pounds} \tag{29}$$

to estimate φ, where $\Omega^T = \Omega$. By choosing Ω, we let φ satisfy the following properties:

① Invariant property. For any $\boldsymbol{X}_0$, we have

$$(\boldsymbol{\pounds} - \boldsymbol{B}\boldsymbol{X}_0)^T \Omega (\boldsymbol{\pounds} - \boldsymbol{B}\boldsymbol{X}_0) = \boldsymbol{\pounds}^T \Omega \boldsymbol{\pounds} \tag{30}$$

Eqn. (30) holds if and only if

$$\Omega \boldsymbol{B} = 0 \tag{31}$$

② Unibiased property. According to the definition of unbiased estimation, we have

$$\boldsymbol{E}(\boldsymbol{\pounds}^T \Omega \boldsymbol{\pounds}) = \sum_{j=1}^{q} \sigma_j^2 \mathrm{tr}(\Omega \boldsymbol{N}_{aj}) = \sum_{j=1}^{q} \sigma_j^2 \alpha_j \tag{32}$$

therefore Eqn. (32) holds for any $\sigma_j^2 (j = 1, 2, \cdots, q)$ if only Ω satisfies

$$\alpha_j = \mathrm{tr}(\Omega \boldsymbol{N}_{aj}) \tag{33}$$

③ Minimum norm property. If $\xi_1, \xi_2, \cdots, \xi_q$ are all known, then the estimator of $\varphi = \alpha^T \theta$ is

$$\frac{\alpha_1 \xi_1^T \xi_1}{n_1} + \frac{\alpha_2 \xi_2^T \xi_2}{n_2} + \cdots + \frac{\alpha_q \xi_q^T \xi_q}{n_q} = \xi^T \Delta \xi \tag{34}$$

where

$$\xi = (\xi_1^T,\ \xi_2^T,\ \cdots,\ \xi_q^T)^T$$

$$\Delta = \mathrm{diag}[\frac{\alpha_1}{\boldsymbol{n}_1}\boldsymbol{I}_1,\ \frac{\alpha_2}{\boldsymbol{n}_2}\boldsymbol{I}_2,\ \cdots,\ \frac{d_q}{\boldsymbol{n}_q}\boldsymbol{I}_q]$$

Now, let $\boldsymbol{\pounds}^T \Omega \boldsymbol{\pounds}$ estimate φ, taking care of the condition of invariance, we have

$$\boldsymbol{\pounds}^T \Omega \boldsymbol{\pounds} = \xi^T \boldsymbol{H}^T \Omega \boldsymbol{H} \xi \tag{35}$$

where

$$\boldsymbol{H} = [\boldsymbol{H}_1,\ \boldsymbol{H}_2,\ \cdots,\ \boldsymbol{H}_q]$$

The principle of minimum norm is

$$\| \boldsymbol{H}\boldsymbol{T} \Omega \boldsymbol{H} - \Delta \|^2 = \min \tag{36}$$

i. e.

$$\| \boldsymbol{H}^T \Omega \boldsymbol{H} - \Delta \|^2 = \mathrm{tr}(\Omega \boldsymbol{T} \Omega \boldsymbol{T}) - \mathrm{tr}\,\Delta^2 = \min \tag{37}$$

where $\boldsymbol{T} = \boldsymbol{H}\boldsymbol{H}^T = \sum_{j=1}^{q} \boldsymbol{H}_j \boldsymbol{H}_j^T = \sum_{j=1}^{q} \boldsymbol{N}_{aj}$.

Eqn. (37) can be rewritten as

$$\mathrm{tr}(\Omega \boldsymbol{T})^2 = \min \tag{38}$$

Let

$$\Omega_0 = \sum_{i=1}^{q} \lambda_i \Omega_i \tag{39}$$

where $\Omega_i = \boldsymbol{M}_0 \boldsymbol{N}_{ai} \boldsymbol{M}_0$ and $\boldsymbol{M}_0 = \boldsymbol{T}^{-1} - \boldsymbol{T}^{-1} \cdot \boldsymbol{B}(\boldsymbol{B}^T \boldsymbol{T}^{-1} \boldsymbol{B})^{-1} \boldsymbol{B}^T \boldsymbol{T}^{-1}$; and let $\Omega' = \Omega - \Omega_0$. It is evident that Ω_0 satisfies (31). In order to let Ω_0 satisfy (33), we can obtain the following condition:

$$\sum_{i=1}^{q} \mathrm{tr}(\boldsymbol{M}_0 \boldsymbol{N}_{ai} \boldsymbol{M}_0 \boldsymbol{N}_{aj}) \lambda_i = \alpha_j \quad (j = 1, 2, \cdots, q) \tag{40}$$

we can prove that if $\Omega' \boldsymbol{B} = 0$, then $\mathrm{tr}(\Omega' \boldsymbol{N}_{aj}) = 0 (j = 1, 2, \cdots q)$. So

$$\mathrm{tr}(\Omega \boldsymbol{T})^2 = \mathrm{tr}[(\Omega_0 + \Omega')\boldsymbol{T}]^2 = \mathrm{tr}(\Omega_0 \boldsymbol{T})^2 + \mathrm{tr}(\Omega' \boldsymbol{T})^2 \geqslant \mathrm{tr}(\Omega_0 \boldsymbol{T})^2 \tag{41}$$

Therefore $\boldsymbol{\pounds}^T \Omega_0 \boldsymbol{\pounds}$ is the minimum norm quadratic unibiased estimate of $\alpha^T \theta$

Let

$$\boldsymbol{S} = [\boldsymbol{S}_{ij}]_{q \times q}, \quad \boldsymbol{S}_{ij} = \mathrm{tr}(\boldsymbol{M}_0 \boldsymbol{N}_{ai} \boldsymbol{M}_0 \boldsymbol{N}_{aj}),$$

$$d = (d_1,\ d_2,\ \cdots,\ d_q)^T,$$

$$d_i = V^T \boldsymbol{A}^T \boldsymbol{M}_0 \boldsymbol{N}_{ai} \boldsymbol{M}_0 \boldsymbol{A} \boldsymbol{V},$$

$$\lambda = (\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_q)^T$$

Eqn. (40) can be rewritten as

$$\boldsymbol{S}\lambda = \alpha \tag{42}$$

Because

$$\boldsymbol{\pounds}^T \Omega_i \boldsymbol{\pounds} = \boldsymbol{V}^T \boldsymbol{A}^T \boldsymbol{M}_0 \boldsymbol{N}_{ai} \boldsymbol{M}_0 \boldsymbol{A} \boldsymbol{V} \quad (i = 1, 2, \cdots, q) \tag{43}$$

$$\boldsymbol{\pounds}^T \Omega \boldsymbol{\pounds} = \sum_{i=1}^{q} \lambda_i \boldsymbol{V}^T \boldsymbol{A}^T \boldsymbol{M}_0 \boldsymbol{N}_{ai} \boldsymbol{M}_0 \boldsymbol{A} \boldsymbol{V} = \boldsymbol{d}^T \lambda = \boldsymbol{d}^T \boldsymbol{S}^{-1} \alpha = \alpha^T \boldsymbol{S}^{-1} \boldsymbol{d} = \alpha^T \hat{\theta} \tag{44}$$

So, MINQUE of $\hat{\theta} = (\hat{\sigma}_1^2,\ \hat{\sigma}_2^2,\ \cdots,\ \hat{\sigma}_q^2)^T$ is

$$\boldsymbol{S}\hat{\theta} = \boldsymbol{d} \tag{45}$$

A general formula (45) of MINQUE is applied to all adjustment models. For example, when $\boldsymbol{C} = 0$, Eqn. (1) becomes the functional model of condition adjustment with unknown parameters. S_{ij} and $\boldsymbol{d}_i$ in (45) are respectively

$$S_{ij} = \mathrm{tr}(\boldsymbol{M}_0 \boldsymbol{N}_{ai} \boldsymbol{M}_0 \boldsymbol{N}_{aj}) \quad (i, j = 1, 2, \cdots, q) \tag{46a}$$

$$\boldsymbol{d}_i = \boldsymbol{\pounds}^T \boldsymbol{M}_0 \boldsymbol{N}_{ai} \boldsymbol{M}_0 \boldsymbol{\pounds} \quad (i = 1, 2, \cdots, q) \tag{46b}$$

where

$$\boldsymbol{M}_0 = \boldsymbol{T}^{-1} - \boldsymbol{T}^{-1} \boldsymbol{B}(\boldsymbol{B}^T \boldsymbol{T}^{-1} \boldsymbol{B})^{-1} \boldsymbol{B}^T \boldsymbol{T}^{-1} \tag{46c}$$

It is just the result derived by Sjöberg (1983)[3]. When $\boldsymbol{B} = 0$ and $\boldsymbol{C} = 0$, Eqn. (1) becomes the functional model of condition adjustment, and Eqn. (46c) is reduced to $\boldsymbol{M}_0 = \boldsymbol{T}^{-1}$.

When $\boldsymbol{A} = -\boldsymbol{I}$, Eqn. (1) becomes the functional model of parameter adjustment with constraints among the parameter. In this case

$$S_{ij} = \mathrm{tr}(\boldsymbol{M}_0 \tilde{\boldsymbol{Q}}_i \boldsymbol{M}_0 \tilde{\boldsymbol{Q}}_j) \quad (i, j = 1, 2, \cdots, q) \tag{47a}$$

$$\boldsymbol{d}_i = \boldsymbol{V}^T \boldsymbol{M}_0 \tilde{\boldsymbol{Q}}_i \boldsymbol{M}_0 \boldsymbol{V} \quad (i = 1, 2, \cdots, q) \tag{47b}$$

$$\boldsymbol{M}_0 = \boldsymbol{Q}^{-1} - \boldsymbol{Q}^{-1} \boldsymbol{B}(\boldsymbol{B}^T \boldsymbol{Q}^{-1} \boldsymbol{B})^{-1} \boldsymbol{B}^T \boldsymbol{Q}^{-1} \tag{47c}$$

when $\boldsymbol{A} = -\boldsymbol{I}$ and $\boldsymbol{C} = 0$, Eqn. (1) becomes the functional model of parameter adjustment, and Eqn. (47c) becomes

$$\boldsymbol{M}_0 = \boldsymbol{Q}^{-1} - \boldsymbol{Q}^{-1} \boldsymbol{B}(\boldsymbol{B}^T \boldsymbol{Q}^{-1} \boldsymbol{B})^{-1} \boldsymbol{B}^T \boldsymbol{Q}^{-1} \tag{48}$$

It is exactly the same as that derived by Rao C R[1], but he thought that $\boldsymbol{L}_1, \boldsymbol{L}_2, \cdots, \boldsymbol{L}_m$ are independent.

4 A UNIVERSAL FORMULA OF ESTIMATION OF VARIANCE AND COVARIANCE COMPONENTS

We take the natural logarithm and derivative of (12) and (15). Letting the derivation equal zero respectively, we find

$$\mathrm{tr}[(\alpha^{\mathrm{T}}\dot{\boldsymbol{N}}_{aa}\alpha^{\mathrm{T}})^{-1}\alpha\boldsymbol{N}_{ai}\alpha^{\mathrm{T}}] = \boldsymbol{\pounds}^{\mathrm{T}}\alpha^{\mathrm{T}}(\alpha\dot{\boldsymbol{N}}_{aa}\alpha^{\mathrm{T}})^{-1}\cdot \alpha\boldsymbol{N}_{ai}\alpha^{\mathrm{T}}(\alpha\dot{\boldsymbol{N}}_{aa}\alpha^{\mathrm{T}})^{-1}\cdot\alpha\boldsymbol{\pounds} \quad (i = 1, 2, \cdots, q) \tag{49}$$

$$\mathrm{tr}(\boldsymbol{MN}_{ai}) = \boldsymbol{\pounds}^{\mathrm{T}}\boldsymbol{MN}_{ai}\boldsymbol{M\pounds} \quad (i = 1, 2, \cdots, q) \tag{50}$$

1) Helmert type

As we all have known, $\boldsymbol{A}\Delta + \boldsymbol{Bx}_1 - \boldsymbol{\pounds} = 0$ and $\alpha\boldsymbol{B} = 0$ with

$$\alpha\boldsymbol{A}\Delta = \alpha\boldsymbol{\pounds} \tag{51}$$

Substituting (51) into (49) gives

$$\mathrm{tr}[(\alpha\dot{\boldsymbol{N}}_{aa}\alpha^{\mathrm{T}})^{-1}\alpha\boldsymbol{N}_{ai}\alpha^{\mathrm{T}}] = \mathrm{tr}[(\alpha\dot{\boldsymbol{N}}_{aa}\alpha^{\mathrm{T}})^{-1}\alpha\boldsymbol{N}_{ai}\cdot \alpha^{\mathrm{T}}(\alpha\dot{\boldsymbol{N}}_{aa}\alpha^{\mathrm{T}})^{-1})\cdot \alpha\boldsymbol{A}\Delta\Delta^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\alpha^{\mathrm{T}}] \quad (i = 1, 2, \cdots, q) \tag{52}$$

Taking $\boldsymbol{D} = \boldsymbol{E}(\Delta\Delta^{\mathrm{T}})$, Eqn. (52) holds for any $i = 1, 2, \cdots, q$. It has been shown that $\Delta\Delta^{\mathrm{T}}$ is the solution of the variance and covariance D in (6). Because Δ is a vector of errors, this gives us a hint that the maximum likelihood estimation of variance and covariance components based on the marginal likelihood or orthogonal complement likelihood is identical with all kinds of estimations started from the defining formula of variance and covariance matrix.

According to the squares adjustment, we have

$$\boldsymbol{V} = \boldsymbol{QA}^{\mathrm{T}}\boldsymbol{M}_0\boldsymbol{A}\Delta \tag{53}$$

The quadratic form is

$$\boldsymbol{V}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{V} = \mathrm{tr}(\boldsymbol{M}_0\dot{\boldsymbol{N}}_{aa}\boldsymbol{M}_0\boldsymbol{A}\Delta\Delta^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}) \tag{54}$$

Taking the expected value on both sides yields

$$\sum_{i=1}^{q}\boldsymbol{E}(\boldsymbol{V}^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{V}) = \sum_{i=1}^{q}\sum_{i=1}^{q}\mathrm{tr}(\boldsymbol{M}_0\boldsymbol{N}_{ai}\boldsymbol{M}_0\boldsymbol{N}_{aj})\,\sigma_i^2 \tag{55}$$

The definition of P_i can be found in (41) of [3]. Using the residuals to replace its expectation, we can obtain

$$\boldsymbol{S}\vartheta = \boldsymbol{d} \tag{56}$$

Eqn. (56) has applied (45) and

$$\boldsymbol{V}^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{V} = \boldsymbol{d}_i \tag{57}$$

Eqn. (57) is found in (44) of Ref. [3].

2) MINQUE and BQUE

Using $\boldsymbol{M}\dot{\boldsymbol{N}}_{aa}\boldsymbol{M} = \boldsymbol{M}$, $\boldsymbol{MB} = 0$ and (50), we have

$$\mathrm{tr}(\boldsymbol{MN}_{ai}) = \mathrm{tr}(\boldsymbol{M}\dot{\boldsymbol{N}}_{aa}MN_{ai}) = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{MN}_{ai}\boldsymbol{MAV} \quad (i = 1, 2, \cdots, q) \tag{58}$$

therefore

$$\boldsymbol{S}\vartheta = \boldsymbol{d} \tag{59}$$

Hence, we may come to the conclusion that Helmert type estimation, MINQUE, BQUE and maximum likelihood estimation are identical with one another. Therefore Eqn. (45) is the universal formula of estimation of variance and covariance components.

5 A UNIVERSAL FORMULA FOR ACCURACY EVALUATION

We have already obtained a universal formula (45) of estimation of variance and covarinace components. But in the estimation of variance and covariance, accuracies in the estimates of individual variance components must be taken into account. Now, we derive the formula for accuracy evaluation.

From (45), we have

$$\vartheta = \boldsymbol{S}^{-1}\boldsymbol{d} \tag{60}$$

Taking the variance on both sides

$$D(\vartheta) = \boldsymbol{S}^{-1}D(\boldsymbol{d})\boldsymbol{S}^{-1} \tag{61}$$

In terms of (37), (39) and (53), we obtain

$$\boldsymbol{V} \sim N(0, \sigma_0^2\boldsymbol{QA}^{\mathrm{T}}\boldsymbol{M}_0\boldsymbol{AQ}) \tag{62}$$

It can easily be proved that

$$\mathrm{cov}(\boldsymbol{V}^{\mathrm{T}}\boldsymbol{YV}, \boldsymbol{V}^{\mathrm{T}}\boldsymbol{ZV}) = 2\mathrm{tr}[\boldsymbol{YD}(\boldsymbol{V})\boldsymbol{ZD}(\boldsymbol{V})] \tag{63}$$

where $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are any symmetric invertible matrices, $\boldsymbol{D}(\boldsymbol{V}) = \sigma_0^2\boldsymbol{QA}^{\mathrm{T}}\boldsymbol{M}_0\boldsymbol{AQ}$. From (42a), we have

$$\mathrm{cov}(\boldsymbol{d}_i, \boldsymbol{d}_j) = 2\sigma_0^4 S_{ij} \quad (i, j = 1, 2, \cdots, q) \tag{64}$$

Eqn. (64) has applied (63) and $\boldsymbol{M}_0\boldsymbol{AQA}^{\mathrm{T}}\boldsymbol{M}_0 = \boldsymbol{M}_0$. Therefore, we can obtain

$$D(\boldsymbol{d}) = 2\sigma_0^4\boldsymbol{S} \tag{65a}$$

$$D(\vartheta) = 2\sigma_0^4\boldsymbol{S}^{-1} \tag{65b}$$

$$D(\hat{\sigma}_i^2) = 2\sigma_0^4\bar{S}_{ii} \quad (i = 1, 2, \cdots, q) \tag{65c}$$

where $\bar{S}_{ii}$ is ith diagonal element of $\boldsymbol{S}^{-1}$. All formulas for accuracy evalution in Ref. [5] are special cases of Eqn. (65). Formula(65c) of accuracy evaluation is applied to all adjustment models. For example, when $\boldsymbol{C} = 0$, Eqn. (11) becomes the functional model of condition adjustment with unknown parameter. S_{ij} in (65) are S_{ij} in (46a). When $\boldsymbol{A} = -\boldsymbol{I}$, $\boldsymbol{C} = 0$, $M = 1$, we have the formula of accuracy evaluation of unit weight variance

$$D(\hat{\sigma}_0^2) = 2\sigma_0^4/(c - u) \tag{66}$$

This is a universal formula we usually use.

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(Edited by PENG Chao-qun)