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Theoretical study on damage bifurcation of unstable failure process of quasi-brittle materials

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Abstract: To obtain the damage effect in the process of elasto-plasticity deformation of quasi-brittle materials, the isotropic damage loading-unloading function and damage variable were introduced to non-continuous bifurcation. The critical bifurcation orientation and its corresponding hardening modulus for quasi-brittle materials were derived considering the effect of stiffness degradation and volumetric dilatancy under the assumption of isotropic damage. The relationships of localized orientation angle and maximal hardening modulus dependent on degree of damage and initial Poisson's ratio of rock were explored. Comparative analyses were conducted to study the bifurcation of uniaxial tension-compression samples under the conditions of plane stress and plane strain. It is shown that as the initial Poisson's ratio or degree of damage increases, the localization orientation angle of the plane uniaxial compression sample tends to be initiated to decrease. However, the localization orientation angle of the plane uniaxial tension sample tends to be initiated to increase. The sum of orientation angle under tension and compression conditions is 90˚. There are plane stress and plane strain cases of the maximum hardening modulus that is independent of the uniaxial compression and tension.

Key words: isotropic damage; orientation of localization; maximum hardening modulus; bifurcation; uniaxial sample

1 Introduction

A bifurcation occurs when a small smooth change made to the parameter values of a system causes a sudden 'qualitative' or topological change in its behaviour[1]. It is characterized by the phenomenon that after the material undergoes a definite amount of equal deformation, suddenly it gets into a stage of deformation when the deformation of high localization takes place. HILL[2], RUDNICKI and RICE[3−4], OTTOSEN and RUNESSON[5], NEILSEN[6] studied on bifurcation and instability of materials from the viewpoint of elasto-plasticity. However, the effects of damage of materials on bifurcation and instability were not taken into consideration. LI and YE[7], ZEND et al[8], ZHAO et al[9] and LÜ et al[10] analyzed the damage localization bifurcation model for rock-like materials under different conditions. LIANG et al[11] gave the characteristics of fractal and percolation of rock subjected to uniaxial compression during their failure process. CHAI et al[12] obtained the characteristics of stability-losing of post-failure rock. In fact, there is also dilatancy of the quasi-brittle material along with presentation of a large quantity of microcosmic defects. At this time, Poisson's ratio is increased and strength and stillness of the material are decreased. Therefore, in analyses of bifurcation and instability of quasi-brittle materials, the two nonlinear deformation features of elasto-plasticity and damage should be taken into consideration simultaneously. In this work, the isotropic damage loading-unloading function and damage variable are introduced into non-continuous bifurcation. The critical bifurcation orientation and its corresponding hardening modulus for quasi-brittle materials are derived considering the effect of stiffness degradation and volumetric dilatancy under the assumption of isotropic damage. The relationships of localized orientation angle and maximal hardening modulus depended on degree of damage and initial Poisson's ratio of rock are explored.

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Comparative analyses are conducted to study the bifurcation of uniaxial tension-compression samples under the conditions of plane stress and plane strain.

2 Theory derivation of damage constitutive functions

Traditionally, the constitutive relationship for isotropic damage may be expressed as

$$\boldsymbol{\sigma}_{ij}=\boldsymbol{D}_{ijkl}\boldsymbol{\varepsilon}_{kl} \tag{1}$$

$$\boldsymbol{C}:\boldsymbol{D}=\boldsymbol{I}_4 \tag{2}$$

where $\boldsymbol{C}$ denotes flexibility tensor of material; $\boldsymbol{I}_4$ is fourth-order unit tensor; $\boldsymbol{\varepsilon}_{kl}$ is total second order strain tensor; $\boldsymbol{D}_{ijkl}=(1-d)\boldsymbol{D}^0_{ijkl}$, is elastic damage modulus of material which is progressively degraded along with the development of microcosmic defects[13].

Their derivatives can be written as

$$\dot{\boldsymbol{D}}_{ijkl}=-\dot{d}\boldsymbol{D}^0_{ijkl},\ \ \boldsymbol{C}_{ijkl}=\hat{d}\boldsymbol{C}^0_{ijkl} \tag{3}$$

where $\boldsymbol{D}^0_{ijkl}$ is initial elastic damage modulus; $\boldsymbol{C}^0_{ijkl}$ denotes initial flexibility tensor of material; and d is the isotropic variable depended only on the plastic strain history.

There is no change of damage modulus in the process of elastic loading and unloading and neutral loading. $\hat{d}=1/(1-d)$. Therefore,

$$\dot{d}=\left(\frac{\partial d}{\partial \boldsymbol{\varepsilon}_{\mathrm{p}}}\right)\cdot\dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} \tag{4}$$

where $\boldsymbol{\varepsilon}_{\mathrm{p}}$ is the characteristic plastic strain.

The damage loading-unloading function is introduced:

$$F(\boldsymbol{\sigma}_{ij},\lambda)=0 \tag{5}$$

where λ is an inner variable in elastic domain. The material will further be damaged and degraded when the inner stress level of the material satisfies yield function F=0. Let G be damage potential function of material, and $\dot{\boldsymbol{\varepsilon}}^{\mathrm{d}}_{ij}$ be damage strain due to the degradation of material. The incremental constitutive relationship of material is

$$\dot{\boldsymbol{\sigma}}_{ij}=\boldsymbol{D}_{ijkl}(\dot{\varepsilon}_{kl}-\dot{\varepsilon}^{\mathrm{d}}_{kl}) \tag{6}$$

When potential function differs from yield function and using non-associated flow rule, it gives

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{d}}_{kl}=\dot{\lambda}\boldsymbol{g}_{kl},\ \ \boldsymbol{g}_{kl}=\frac{\partial G}{\partial \boldsymbol{\sigma}_{kl}} \tag{7}$$

where $\boldsymbol{g}_{kl}$ is gradient function of damage potential function G in stress domain.

According to the identity condition of plasticity, the rate-format of Eq.(5) is

$$\dot{F}=\frac{\partial F}{\partial \boldsymbol{\sigma}_{ij}}\dot{\boldsymbol{\sigma}}_{ij}+\frac{\partial F}{\partial \lambda}\dot{\lambda}=\boldsymbol{f}_{ij}\dot{\boldsymbol{\sigma}}_{ij}-H\dot{\lambda}=0 \tag{8}$$

where $\boldsymbol{f}_{ij}=\frac{\partial F}{\partial \boldsymbol{\sigma}_{ij}}$, $H=-\frac{\partial F}{\partial \lambda}$, are hardening and softening moduli, respectively; and $\dot{\lambda}$ is damage gene.

Eq.(2) may be expressed by rate format as

$$\dot{\boldsymbol{D}}_{ijkl}=-\boldsymbol{D}_{ijpq}\dot{\boldsymbol{C}}_{pqrs}\boldsymbol{D}_{rskl} \tag{9}$$

By introducing flexibility tensor, the flow rule is expressed as

$$\dot{\boldsymbol{C}}_{ijkl}=\dot{\lambda}\boldsymbol{M}_{ijkl},\ \ \dot{\boldsymbol{\varepsilon}}^{\mathrm{d}}_{pq}=\dot{\boldsymbol{C}}_{pqrs}\boldsymbol{\sigma}_{rs} \tag{10}$$

where $\boldsymbol{M}_{ijkl}$ is the initial flexibility tensor.

From Eqs.(7) and (10), the following equation is attained:

$$\boldsymbol{g}_{ij}=\boldsymbol{M}_{ijkl}\boldsymbol{\sigma}_{kl} \tag{11}$$

From Eqs.(6), (7) and (8), the following equations are attained:

$$\dot{\lambda}=\frac{1}{A}\boldsymbol{f}_{ij}\boldsymbol{D}_{ijkl}\dot{\varepsilon}_{kl} \tag{12}$$

$$A=H+\boldsymbol{f}_{pq}\boldsymbol{D}_{pqrs}\boldsymbol{M}_{rsuv}\boldsymbol{\sigma}_{uv}>0 \tag{13}$$

Substitution of Eqs.(12) and (13) into Eqs.(6) and (7), the isotropic damage constitutive relation in incremental form can be written as

$$\dot{\boldsymbol{\sigma}}_{ij}=\boldsymbol{D}^{\mathrm{ed}}_{ijkl}\dot{\boldsymbol{\varepsilon}}_{ij} \tag{14}$$

where $\boldsymbol{D}^{\mathrm{ed}}_{ijkl}$ is elastic-damage tangential modulus matrix.

$$\boldsymbol{D}^{\mathrm{ed}}_{ijkl}=\begin{cases}\boldsymbol{D}_{ijkl}, & \text{Elastic loading}\\ \boldsymbol{D}_{ijkl}-\frac{1}{A}\boldsymbol{D}_{ijkl}\boldsymbol{M}_{abcd}\boldsymbol{\sigma}_{cd}\boldsymbol{f}_{xy}\boldsymbol{D}_{xykl}, & \text{Damage loading}\end{cases} \tag{15}$$

The damage loading-unloading criterion follows Kuhn-Tucker reciprocal condition:

$$\dot{F}\le 0,\ \dot{\lambda}\ge 0,\ \dot{F}\dot{\lambda}=0 \tag{16}$$

where damage gene, $\dot{\lambda}=\dot{\hat{d}}=\dot{d}/(1-d)^2$, and initial flexibility tensor, $\boldsymbol{M}_{ijkl}=\boldsymbol{C}^0_{ijkl}$, are proposed by CAROL et al[14].

Then $\boldsymbol{g}_{ij}$ of Eq.(11) can be rewritten as

$$\boldsymbol{g}_{ij}=\boldsymbol{C}^0_{ijkl}\boldsymbol{\sigma}_{kl}=\boldsymbol{\varepsilon}^0_{ij} \tag{17}$$

Thus, the general expression of hardening modulus is

$$H=-\frac{\partial F}{\partial \lambda}=-\frac{\partial f(w^0,\hat{d})}{\partial \hat{d}}+\frac{\partial r(\hat{d})}{\partial \hat{d}} \tag{18}$$

And the gradient function of loading function is

$$f_{ij}=\frac{\partial F}{\partial \boldsymbol{\sigma}_{ij}}=\frac{\partial f(w^0,\hat{d})}{\partial w^0}\boldsymbol{C}^0_{ijkl}\boldsymbol{\sigma}_{kl}=\frac{\partial f(w^0,\hat{d})}{\partial w^0}\varepsilon^0_{ij} \tag{19}$$

where damage loading-unloading function, $F=f(w^0,\hat{d})-r(\hat{d})$, is proposed by CAROL et al[14]; $w^0=\frac{1}{2}\boldsymbol{\sigma}_{ij}\boldsymbol{C}^0_{ijkl}\cdot\boldsymbol{\sigma}_{kl}$ and $f(w^0,\hat{d})$ are equivalent extending and driving force of cracks, respectively. $r(\hat{d})$ is representative radius of yield plane of damage. This denotes that the cracking and yielding of material depend on current accumulative damage.

Substituting Eqs.(17)−(19) into Eq.(13), and considering Eq.(3), it gives

$$A=-\frac{\partial f(w^0,\hat{d})}{\partial \hat{d}}+\frac{\partial r(\hat{d})}{\partial \hat{d}}+2(1-d)w^0\frac{\partial f(w^0,\hat{d})}{\partial w^0} \tag{20}$$

Finally, the elasto-plastic damage tangential modulus is expressed as

$$\boldsymbol{D}^{\mathrm{ed}}_{ijkl}=(1-d)\boldsymbol{D}^0_{ijkl}-\frac{(1-d)^2(\partial f/\partial w^0)}{\frac{\partial r}{\partial \hat{d}}-\frac{\partial f}{\partial \hat{d}}+2(1-d)w^0\frac{\partial f}{\partial w^0}}\boldsymbol{\sigma}_{ij}\boldsymbol{\sigma}_{kl} \tag{21}$$

3 Damage localized bifurcation

The condition of non-continuous bifurcation of damaged material is depended on singularity of the following damage localized tensor $\boldsymbol{Q}^{\mathrm{d}}_{il}$, and $\boldsymbol{Q}^{\mathrm{d}}_{il}$ is defined as

$$\boldsymbol{Q}^{\mathrm{d}}_{il}=\boldsymbol{n}_j\boldsymbol{D}^{\mathrm{ed}}_{ijkl}\boldsymbol{n}_k \tag{22}$$

where $\boldsymbol{n}_j$ (j=1, 2, 3) is outer unit normal vector of characteristic plane of localized zone. The necessary condition when non-continuous bifurcation appears in the material is

$$\det[\boldsymbol{Q}^{\mathrm{d}}_{il}]=0 \tag{23}$$

The condition can be regarded as necessary condition that nonlinear uniform algebraic function of unit normal vector of characteristic plane of localization obtains non singular solution. Therefore, corresponding algebraic functions of uniform linearity should be considered.

Thus, the following eigen-value problem is to be considered:

$$\boldsymbol{Q}^{\mathrm{d}}_{il}y^{(i)}_l=\lambda^{(i)}\boldsymbol{Q}^{\mathrm{d}}_{il}y^{(i)}_l\quad (i=1, 2, 3) \tag{24}$$

Considering that $\boldsymbol{D}^{\mathrm{ed}}_{ijkl}$ is symmetric and positively definite, it can be proved that $\boldsymbol{Q}^{\mathrm{d}}_{il}$ is positive as well. Its inverse matrix is $(\boldsymbol{Q}^{\mathrm{d}}_{il})^{-1}=\boldsymbol{P}^{\mathrm{d}}_{il}$. So, Eq.(24) may be rewritten as

$$\boldsymbol{B}_{jl}y^{(i)}_l=\lambda^{(i)}y^{(i)}_j\quad (i=1, 2, 3) \tag{25}$$

where $\boldsymbol{B}_{jl}=\boldsymbol{P}^{\mathrm{d}}_{ji}\boldsymbol{Q}^{\mathrm{d}}_{il}=\delta_{jl}-\frac{1}{A}\boldsymbol{P}^{\mathrm{d}}_{ji}b_i a_l$, $a_l=\boldsymbol{f}_{mn}\boldsymbol{D}^0_{mnkl}\boldsymbol{n}_k$, $b_i=\boldsymbol{n}_j\boldsymbol{D}^0_{ijst}\boldsymbol{g}_{st}$.

Considering $\boldsymbol{P}^{\mathrm{d}}_{il}$ and $\boldsymbol{Q}^{\mathrm{d}}_{il}$ are non-singular, and the singularity of $\boldsymbol{B}_{jl}$ is nonsingular, it is indicated that λ=1 is an eigen-value with a multiplicity of two for the eigen-value problem in Eq.(25). Associated this with the properties of matrix, the third eigen-value can be attained as

$$\lambda^{(3)}=1-\frac{1}{A}\boldsymbol{a}_i\boldsymbol{P}^{\mathrm{e}}_{ij}\boldsymbol{b}_j \tag{26}$$

To make $\boldsymbol{B}_{ji}$ singular, at least one eigen-value should be equal to zero. Then $\lambda^{(3)}$=0. Considering Eq.(13), the hardening modulus in accordance with occurrence of localization can be attained as

$$H=-\boldsymbol{f}_{ij}\boldsymbol{D}_{ijkl}\boldsymbol{g}_{kl}+\boldsymbol{n}_j\boldsymbol{D}_{ijst}\boldsymbol{g}_{st}\boldsymbol{P}^{\mathrm{d}}_{il}\boldsymbol{f}_{mn}\boldsymbol{D}_{mnkl}\boldsymbol{n}_k \tag{27}$$

Assuming $\boldsymbol{D}^0_{ijkl}=\boldsymbol{D}^{\mathrm{e}}_{ijkl}$, where $\boldsymbol{D}^{\mathrm{e}}_{ijkl}$ is isotropic elastic tensor of material, from Eq.(3) it gives

$$\boldsymbol{D}_{ijkl}=2G^{\mathrm{d}}\left[\frac{\nu^{\mathrm{d}}}{1-2\nu^{\mathrm{d}}}\boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{kl}+\frac{1}{2}(\boldsymbol{\delta}_{ik}\boldsymbol{\delta}_{jl}+\boldsymbol{\delta}_{il}\boldsymbol{\delta}_{jk})\right] \tag{28}$$

where $G^{\mathrm{d}}=(1-d)G$ and $\nu^{\mathrm{d}}=\frac{3\nu_0+(1-2\nu_0)d}{3-(1-2\nu_0)d}$ are post-damage shear modulus and Poisson's ratio, respectively, of material[14]. G and ν_0 are prior-damage shear modulus and Poisson's ratio, respectively, of material.

$$\begin{cases}\boldsymbol{Q}^{\mathrm{d}}_{il}=G^{\mathrm{d}}(\frac{1}{1-2\nu^{\mathrm{d}}}\boldsymbol{n}_i\boldsymbol{n}_l+\boldsymbol{\delta}_{il})\\ \boldsymbol{P}_{il}=\frac{1}{G^{\mathrm{d}}}(-\frac{1}{2(1-\nu^{\mathrm{d}})}\boldsymbol{n}_i\boldsymbol{n}_l+\boldsymbol{\delta}_{il})\end{cases} \tag{29}$$

Among the orientations along which localization is possible to occur, only the orientation corresponding to the maximum hardening modulus firstly satisfies the condition of localization, so it is the orientation along which localization occurs first. Therefore, the orientation that makes hardening modulus attain the maximum value $H^{\mathrm{d}}=\max H(\boldsymbol{n}_i)$ is the critical localized orientation.

Arrangement of the above equations reaches the following expression:

$$\begin{aligned}\frac{H}{2G^{\mathrm{d}}}=&2\boldsymbol{n}_k\boldsymbol{f}_{kl}\boldsymbol{g}_{ij}\boldsymbol{n}_j+\\&\frac{\nu^{\mathrm{d}}}{1-\nu^{\mathrm{d}}}\left[\boldsymbol{n}_i(\boldsymbol{g}_{ss}\boldsymbol{f}_{ij}+\boldsymbol{f}_{ss}\boldsymbol{g}_{ij})\boldsymbol{n}_j-\boldsymbol{f}_{ii}\boldsymbol{g}_{ss}\right]-\\&\boldsymbol{f}_{ij}\boldsymbol{g}_{ij}-\frac{1}{1-\nu^{\mathrm{d}}}\boldsymbol{n}_i\boldsymbol{f}_{ij}\boldsymbol{n}_j\boldsymbol{n}_k\boldsymbol{g}_{kl}\boldsymbol{n}_l\end{aligned} \tag{30}$$

where $\boldsymbol{f}_v=\boldsymbol{f}_{ii}$; $\bar{\boldsymbol{f}}_{ij}=\boldsymbol{f}_{ij}-\frac{1}{3}\boldsymbol{\delta}_{ij}\boldsymbol{f}_v$; $\boldsymbol{g}_v=\xi\boldsymbol{f}_v$; $\bar{\boldsymbol{g}}_{ij}=\xi\bar{\boldsymbol{f}}_{ij}$.

$\boldsymbol{f}_v$, $\boldsymbol{f}_{ij}$ and $\boldsymbol{g}_v$, $\boldsymbol{g}_{ij}$ are the first invariables and deviatoric tensors of gradient function of yield function and damage function, respectively; $\xi=[\partial f(w^0,\hat{d})/\partial w^0]^{-1}$. Assuming $\bar{\boldsymbol{f}}_i$ (i=1, 2, 3) to be the main values of $\bar{\boldsymbol{f}}_{ij}$ and

$$r=\varphi(\boldsymbol{f}_v+\boldsymbol{g}_v)\,,\quad k=\frac{1}{2}\sum_{i=1}^{3}\bar{\boldsymbol{f}}_i^2+\frac{2}{3}\varphi\boldsymbol{f}_v^2 \tag{31}$$

Substituting above equations into Eq.(30), the unitary hardening modulus can be rewritten as

$$\frac{H}{4\xi G^{\mathrm{d}}}=\sum_{m=1}^{3}\left(\bar{f}_m^2+r\bar{f}_m\right)n_m^2-\psi\left(\sum_{l=1}^{3}\bar{\boldsymbol{f}}_l\boldsymbol{n}_l^2\right)^2-k \tag{32}$$

φ and ψ in Eqs.(31) and (32) are

$$\varphi=\frac{1+v^{\mathrm{d}}}{6(1-v^{\mathrm{d}})},\quad \psi=\frac{1}{2(1-v^{\mathrm{d}})} \tag{33}$$

respectively.

Considering geometric restrained conditions of unit vector $\sum_{j=1}^{3}\boldsymbol{n}_j^2=1$, to obtain the maximum value of hardening modulus for the extremum problem with restrained conditions, the Lagrange's multiplier method is employed to establish the extremum problem without restrained conditions:

$$L=\frac{H}{4\xi G^{\mathrm{d}}}-\beta\left(\sum_{j=1}^{3}\boldsymbol{n}_j^2-1\right) \tag{34}$$

where β is the Lagrangian multiplier to be determined.

The extremum of hardening modulus H in Eq.(34) is to be determined by the following extremum conditions:

$$\frac{\partial L}{\partial \boldsymbol{n}_i}=2A_i\boldsymbol{n}_i=0,\quad \frac{\partial L}{\partial \beta}=-\left(\sum_{j=1}^{3}\boldsymbol{n}_j^2-1\right)=0 \tag{35}$$

where

$$A_i=\bar{f}_i^2+r\bar{f}_i-2\psi m\bar{f}_i-\beta\,,\quad m=\sum_{i=1}^{3}\bar{\boldsymbol{f}}_i\boldsymbol{n}_i^2 \tag{36}$$

The symmetric Hessian $\boldsymbol{H}_{ij}$ of L is

$$\boldsymbol{H}_{ij}=\frac{\partial^2 L}{\partial \boldsymbol{n}_i\partial \boldsymbol{n}_j}=\begin{bmatrix} H_{11} & H_{12} & H_{13}\\ H_{21} & H_{22} & H_{23}\\ H_{31} & H_{32} & H_{33}\end{bmatrix} \tag{37}$$

where $H_{ij}=2A_i\delta_{ij}-8\psi\bar{f}_i\bar{f}_j\boldsymbol{n}_i\boldsymbol{n}_j$ (i=1, 2, 3).

Considering $n_i\neq 0$, it can be seen from the first condition of Eq.(36) that $A_i\equiv 0$, i.e.

$$\begin{bmatrix} \bar{f}_1^2 & \bar{f}_1\bar{f}_2 & \bar{f}_1\bar{f}_3\\ \bar{f}_1\bar{f}_2 & \bar{f}_2^2 & \bar{f}_2\bar{f}_3\\ \bar{f}_1\bar{f}_3 & \bar{f}_2\bar{f}_3 & \bar{f}_3^2\end{bmatrix}\begin{bmatrix} n_1^2\\ n_2^2\\ n_3^2\end{bmatrix}=\frac{1}{2\psi}\begin{bmatrix} \bar{f}_1^2+r\bar{f}_1-\beta\\ \bar{f}_2^2+r\bar{f}_2-\beta\\ \bar{f}_3^2+r\bar{f}_3-\beta\end{bmatrix} \tag{38}$$

If $\boldsymbol{h}_i=[\bar{f}_1n_1\quad \bar{f}_2n_2\quad \bar{f}_3n_3]$, it gives $\boldsymbol{H}_{ij}=-8\psi\boldsymbol{h}_i^{\mathrm{T}}\boldsymbol{h}_j$.

Assuming $\bar{f}_1\geq\bar{f}_2\geq\bar{f}_3$ and $\bar{f}_1\neq 0$, it can be obtained from Eq.(36) that

$$\beta(\bar{f}_1-\bar{f}_2)=-\bar{f}_1\bar{f}_2(\bar{f}_1-\bar{f}_2) \tag{39}$$

$$\beta(\bar{f}_1-\bar{f}_3)=-\bar{f}_1\bar{f}_3(\bar{f}_1-\bar{f}_3) \tag{40}$$

1) When $\bar{f}_1>\bar{f}_2>\bar{f}_3$, Eq.(39) and Eq.(40) are not able be satisfied simultaneously. For this case, no maximum of Eq.(34) exists.

2) When $\bar{f}_1=\bar{f}_2>\bar{f}_3$, for this case, from Eqs.(39) and (40), obviously $\beta=-\bar{f}_1\bar{f}_3$. When $\bar{f}_1\neq 0$ and $\bar{f}_3\neq 0$, from assumptions before and Eq.(38), the following equation is obtained:

$$\bar{f}_1(n_1^2+n_2^2)+\bar{f}_3n_3^2=\frac{1}{2\psi}(\bar{f}_1+\bar{f}_3+r) \tag{41}$$

From Eq.(41) and to associate with the equation $n_1^2+n_2^2=1-n_3^2$, it gives

$$n_3^2=-\frac{\bar{f}_3+(1-2\psi)\bar{f}_1+r}{2\psi(\bar{f}_1-\bar{f}_3)} \tag{42}$$

As $0\leq n_3^2\leq 1$, the following equations are obtained:

$$\bar{f}_3+(1-2\psi)\bar{f}_1+r\leq 0,\quad \bar{f}_1+(1-2\psi)\bar{f}_3+r\geq 0 \tag{43}$$

3) When $\bar{f}_1>\bar{f}_2=\bar{f}_3$, for this case, $\beta=-\bar{f}_1\bar{f}_3$ as well, then

$$n_1^2=-\frac{\bar{f}_1+(1-2\psi)\bar{f}_3+r}{2\psi(\bar{f}_1-\bar{f}_3)},\quad n_2^2+n_3^2=1-n_1^2 \tag{44}$$

$$\bar{f}_3+(1-2\psi)\bar{f}_1+r\leq 0,\quad \bar{f}_1+(1-2\psi)\bar{f}_3+r\geq 0 \tag{45}$$

The expression of maximum hardening modulus for occurrence of non-continuous localization bifurcation can be derived from the above equations as

$$H^{\mathrm{db}}=G^{\mathrm{d}}\xi[\psi(\bar{f}_1+\bar{f}_3+r)^2-\bar{f}_1\bar{f}_3-k] \tag{46}$$

As shown in Fig.1, if $r\leq 0$ and $\bar{f}_1<\bar{f}_2<\bar{f}_3$ and assuming n_2=0, the orientation angle θ_{cr} along which localized damage occurs may be represented by the angle formed by the outer normal vector $\boldsymbol{n}$ of localized characteristic plane and the direction of x_1 axis, and is given by

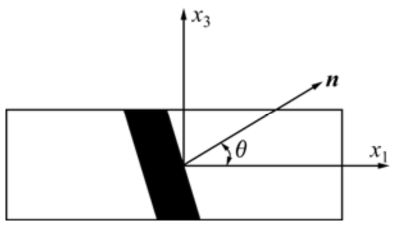

Fig.1 Orientation angle of localization for plane problems

$$\tan^2\theta_{\rm cr}=\frac{\bar{f}_1+(1-2\psi)\bar{f}_3+2\varphi\boldsymbol{f}_v}{\bar{f}_3+(1-2\psi)\bar{f}_1+2\varphi\boldsymbol{f}_v} \tag{47}$$

4 Analysis of localization of uniaxial sample

The theories of analyses on isotropic damage localized bifurcation illustrated above are used to concretely calculate and analyze the localization of uniaxial compression-tension testing samples under plane strain and plane stress conditions. The damage loading and unloading function $F=w^0-r[\hat{d}]$ proposed by JU[15] is used in the following analysis. The results are shown in Table 1.

The dependence of orientation angle of the normal vector of localized characteristic plane on degree of damage and initial Poisson's ratio are obtained by calculations, as shown in Figs.2 and 3, respectively.

1) Whether under plane stress or strain condition, variation of orientation angle of localization is depended on initial Poisson's ratio and degree of damage. Under the condition of uniaxial compression, the larger the initial Poisson's ratio or the degree of damage is, the smaller the orientation angle of localization is, and the slower the tendency of progressive decrease of orientation angle becomes. Under the condition of uniaxial tension, the larger the initial Poisson's ratio or the degree of damage is, the larger the orientation angle of localization is, and the slower the tendency of progressive increase of orientation angle becomes. When d=1 or ν=0.5, the orientation angle keeps unchanged and reaches the minimum value under the condition of uniaxial compression.

2) When d=1 or ν=0.5, the orientation angle keeps unchanged and reaches the maximum value under the condition of uniaxial tension. Under plane strain uniaxial compression condition, the orientation angle of localization $\theta_{\rm cr}$=45°; and under plane stress uniaxial compression condition, the orientation angle of localization $\theta_{\rm cr}$=54.7°; under plane strain uniaxial tension condition, the orientation angle of localization $\theta_{\rm cr}$=45°; and under plane stress uniaxial tension condition, the orientation angle of localization $\theta_{\rm cr}$=35°.

The dependence of maximum hardening modulus on degree of damage and initial Poisson's ratio are obtained by calculations, as shown in Fig.4. It can be seen that:

1) Whether under plane stress or plane strain condition, when $\nu>0.35$, the maximum hardening modulus decreases and reaches zero with the increase of degree of damage. The smaller the initial Poisson's ratio is, the slower the tendency of progressive decrease of maximum hardening modulus becomes. When $\nu\le 0.35$, at the beginning, maximum hardening modulus increases, then decreases and reaches zero with the increase of degree of damage. The smaller the initial Poisson's ratio is, the smaller the maximum hardening modulus becomes.

2) When $\nu<0.20$, the extension of effective value of the degree of damage is different with the change of initial poisson's ration. The larger the initial poisson's ratio is, the large the extension of effective value is. Under different degrees of damage, the effective extension of Poisson's ratio is different. The larger the degree of damage is, the larger the extension is. When ν=0.50, the maximum hardening modulus keeps unchanged and reaches zero.

For comparing characteristic of bifurcation under plane strain and plane stress condition simultaneously, the change relation of maximum hardening modulus and orientation angle of the normal vector of localized characteristic plane on degree of damage under two conditions are given in Figs.5−8 for ν_0=0.3 or d=0.5. It can be seen that the declining rate of localized orientation angle is lower under compression condition

Table 1 Result statistics of damage localization of uniaxial sample under plane condition

Parameters localization	Damage loading and unloading function F	Plane loading conditions			
		Plane strain		Plane stress	
		Uniaxial tension	Uniaxial compression	Uniaxial tension	Uniaxial compression
Orientation of localization	$w^0-r(\hat{d})$	$\arctan\sqrt{\frac{3\nu_0+(1-2\nu_0)d}{3(1-\nu_0)-2d(1-2\nu_0)}}$	$\arctan\sqrt{\frac{3(1-\nu_0)-2d(1-2\nu_0)}{3\nu_0+(1-2\nu_0)d}}$	$\arctan\sqrt{\frac{3\nu_0+(1-2\nu_0)d}{3-(1-2\nu_0)d}}$	$\arctan\sqrt{\frac{3-(1-2\nu_0)d}{3\nu_0+(1-2\nu_0)d}}$
Maximum hardening modulus	$w^0-r(\hat{d})$	$G^{\rm d}\left[\frac{(\nu^{\rm d})^3+(\nu^{\rm d})^2-5\nu^{\rm d}+1}{18(\nu^{\rm d}-1)}\right]\varepsilon_1^2$	$G^{\rm d}\left[\frac{2(\nu^{\rm d})^3+6(\nu^{\rm d})^2+2\nu^{\rm d}-1}{18(\nu^{\rm d}+1)^2}\right]\varepsilon_1^2$	$G^{\rm d}\left[\frac{(\nu^{\rm d})^3+(\nu^{\rm d})^2-5\nu^{\rm d}+1}{18(\nu^{\rm d}-1)}\right]\varepsilon_1^2$	$G^{\rm d}\left[\frac{2(\nu^{\rm d})^3+6(\nu^{\rm d})^2+2\nu^{\rm d}-1}{18(\nu^{\rm d}+1)^2}\right]\varepsilon_1^2$

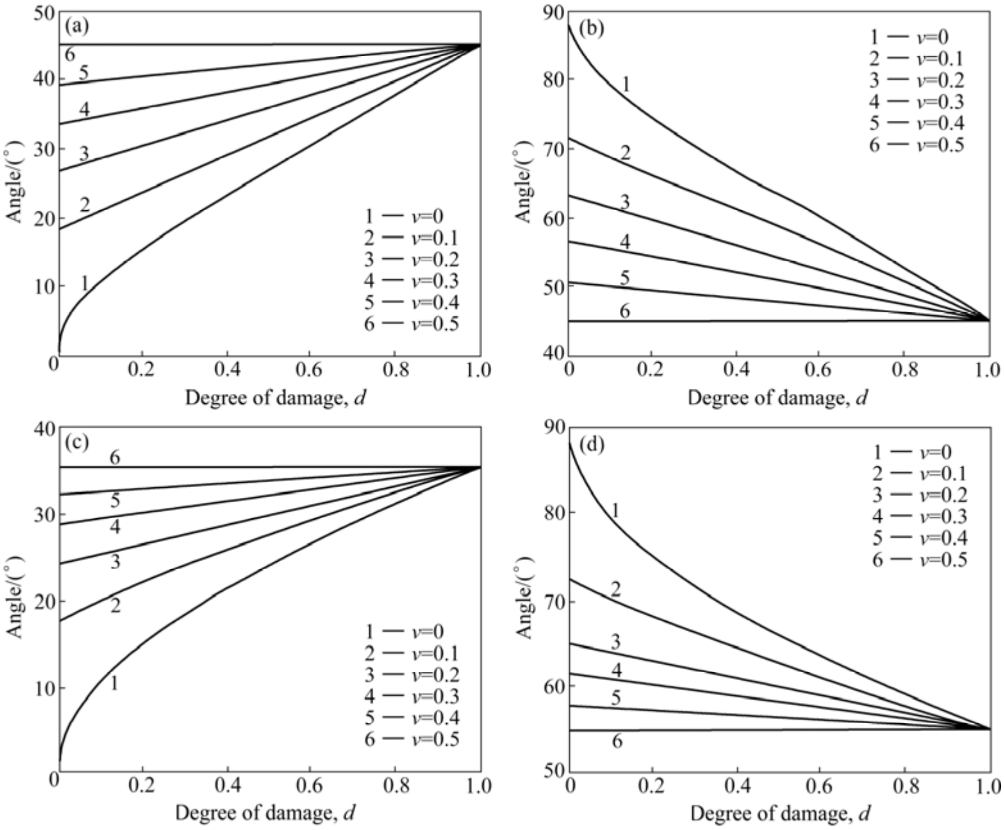

Fig.2 Dependency of orientation of localization on degree of damage for different Poisson's ratios: (a) Plane-strain uniaxial tension; (b) Plane-strain uniaxial compression; (c) Plane-stress uniaxial tension; (d) Plane-stress uniaxial compression

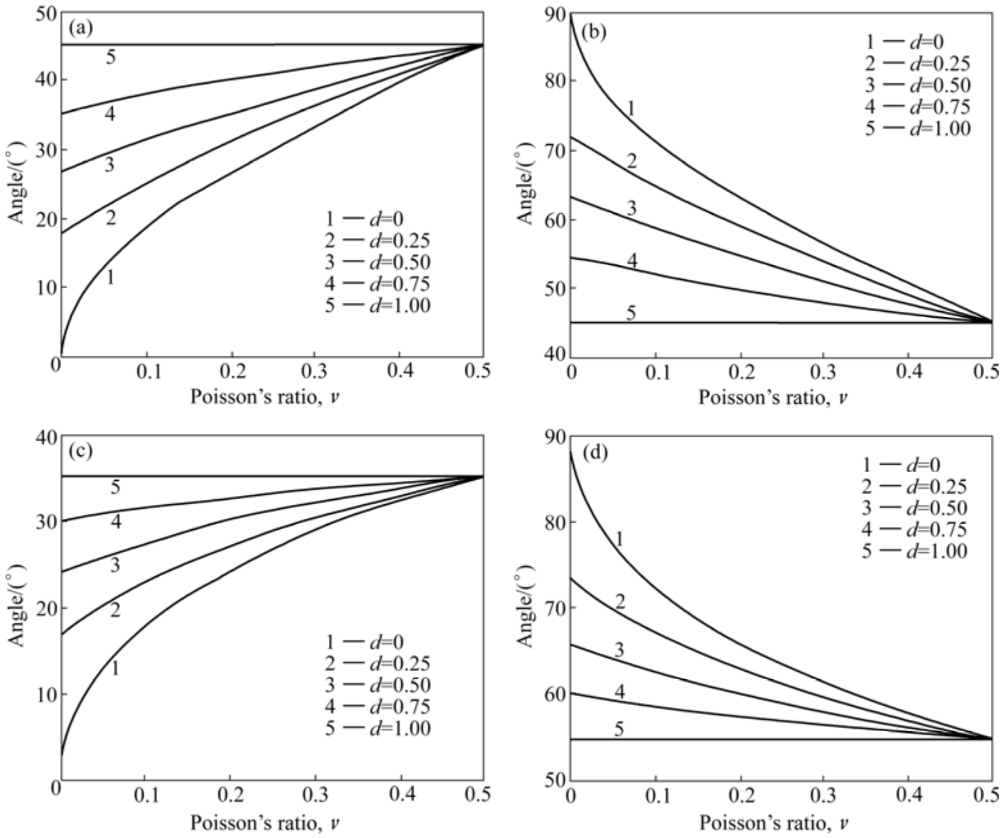

Fig.3 Dependency of orientation of localization on Poisson's ratios for different degrees of damage: (a) Plane-strain uniaxial tension; (b) Plane-strain uniaxial compression; (c) Plane-stress uniaxial tension; (d) Plane-stress uniaxial compression

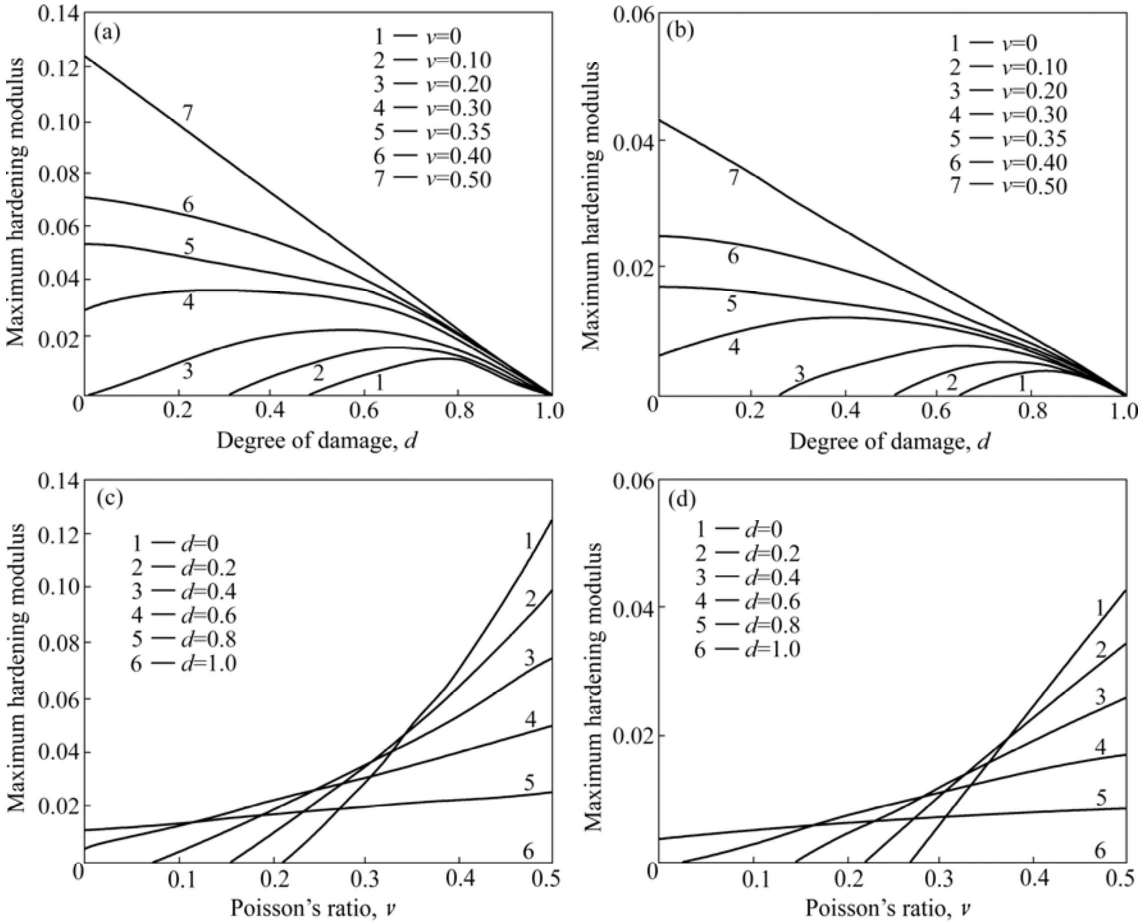

Fig.4 Dependency of maximum hardening modulus ratio on damage degree and Poisson's ratio: (a) Plane strain condition; (b) Plane stress condition; (c) Plane strain condition; (d) Plane stress condition

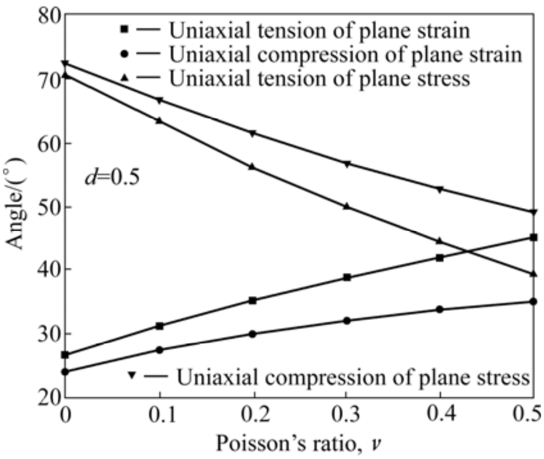

Fig.5 Comparison of dependency of localization orientation on Poisson's ratio under different conditions

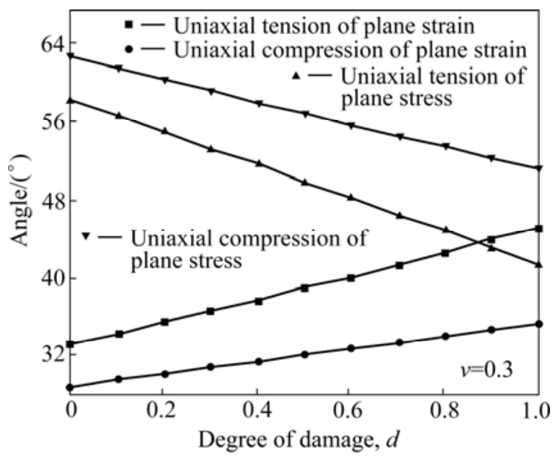

Fig.6 Comparison of dependency of localization orientation on degree of damage under different conditions

than that under tension condition. And the localized orientation angle will be larger with the same degree of damage or initial Poisson's ratio. Under condition of plane strain, the localized orientation angle will increase, if initial Poisson's ratio or the degree of damage increases. Under tension condition, the increasing rate of localized orientation angle is higher than that under compression condition. And the localized orientation angle will increase with the same degree of damage or initial Poisson's ratio. The sum of localized orientation angle under tension and compression conditions is 90°. Secondly, there are plane stress and plane strain cases of the maximum hardening modulus that is independent of compression and tension. Maximum hardening modulus will increase when degree of damage or initial Poisson's ratio increases. Under plane stain condition, the increasing rate of maximum hardening modulus is higher than that under plane stress condition. And maximum hardening modulus will increase with the same degree of damage or initial Poisson's ratio, so does the effective range of Poisson's ratio.

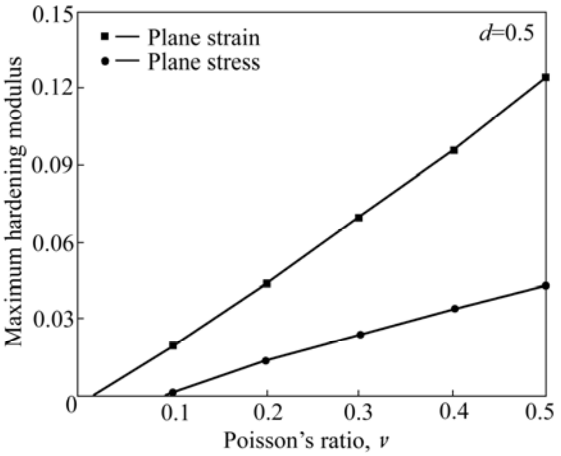

Fig.7 Comparison of dependency of maximum hardening modulus on Poisson's ratio under different conditions

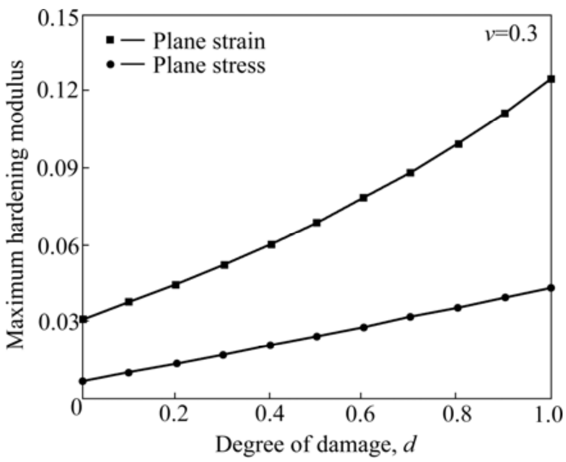

Fig.8 Comparison of dependency of maximum hardening modulus on degree of damage under different conditions

5 Conclusions

Taking damage degradation and dilatancy into consideration, the maximum hardening modulus and orientation angle at occurrence of bifurcation and instability of rock-like materials are deduced.

1) The orientation angle of localization at occurrence of instability is depended on degree of damage and initial Poisson's ratio. The sum of localized orientation angle under tension and compression conditions is 90˚.

2) There are plane stress and plane strain cases of the maximum hardening modulus that is independent of the uniaxial compression and tension.

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