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Closing law and stress intensity factor of elliptical crack under compressive loading ①

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[Abstract] The solution of surface displacement of an elliptical crack under compressive-shear loading was obtained by using the complex function method. The closing mode was established by analyzing the geometrical condition of closing crack, and the corresponding critical stress was solved. The result corrects the traditional viewpoint, in which there exist only open or close states for an elliptical crack, and points out that the local closing is also one of crack states. Based on them, the effect of the closed crack on stress intensity factor was discussed in detail, and its rational formulae are put forward.

[Key words] elliptical crack; compressive loading; closing model; stress intensity factor
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1 INTRODUCTION

It is important to study the fracture law of the internal crack in rock under compressive loading. The crack subjected to compressive-shear stress is often in company with friction, which has great effect on the extension of crack[1~3] and the stress intensity factor[4, 5]. However, up to now, these effects are not clear in rock mechanics circles, which lead to several different mechanism model. For example, someone thought that friction does not act on fracture[6], but, another thought that friction governs fracture[7]; someone thought that there is no the compressive stress intensity factor when crack closes[8], but, another thought that there exists the compressive stress intensity factor when crack closes, who is equal to the negative value of tension stress intensity factor[9]. In author's opinion, these different viewpoints mainly originate from the incomprehension to the closing law of crack. For example, the traditional viewpoints thought that there is only open or close state for an elliptical crack, not transition process between two states, and neglected the contribution of compressive stress intensity factor to the extension of crack[10].

In this paper, the closing law of an elliptical crack is studied in detail by using the compressive-shear element model of an oblique crack under compressive loading. It is proved that the closing mode of crack is governed by shear stress. Based on it, the expression of stress intensity factor at the tip of crack for various closing mode are deduced.

2 SURFACE DISPLACEMENT SOLUTION OF ELLIPTICAL CRACK UNDER COMPRESSIVE LOADING

The closing problem of an oblique elliptical crack under biaxial compressive loading is shown in Fig. 1.

Suppose σ_c be the critical compressive stress which makes the crack close, then by McClintock-Walsh model[11], the normal stress and friction acting on the surface of the closing crack are expressed as

$$\sigma_n = \sigma_y - \sigma_c \quad (1)$$

$$\tau_f = \mu\sigma_n \quad (2)$$

where

$$\sigma_y = \sigma_1 \sin^2\beta + \sigma_2 \cos^2\beta \quad (3)$$

$$\tau_{xy} = -\frac{1}{2}(\sigma_1 - \sigma_2)\sin^2\beta \quad (4)$$

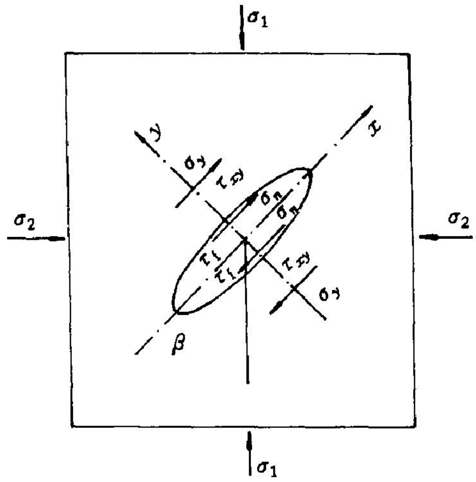

Fig. 1 Oblique elliptical crack under biaxial compressive loading

Because τ_{xy} is negative value in the coordinate systems of Fig. 1, τ_f must be in the direction shown in Fig. 1. According to the superimposition principle, if superimposing an stress system of uniform tension stress $-\sigma_n$ and uniform shear stress τ_f on the stress state in element of Fig. 1, a new crack problem with zero stress at the surface of crack, and normal stress $\sigma_y - \sigma_n = \sigma_c$ and shear stress $\tau_{xy} + \tau_f = \tau_{xy} + \mu(\sigma_y - \sigma_c)$ at infinity will be obtained, as shown in Fig. 2.

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Because the uniform stress system in Figs. 2(b) and (c) do not bring the displacement to surface of the crack, the stress system in Fig. 2(a) is equal to the one in Fig. 2(d).

In order to solve the displacement of an elliptical crack in infinite body under compressive-shear loading, the angle-keep transformation is used to transform the outside area of an ellipsoid in z plane to the outside area of a unit circle in ω plane.

$$z = \omega(\zeta) = R(\zeta + m/\zeta) \tag{5}$$

where $\zeta = \rho e^{i\theta}$, $R = (a+b)/2$, $m = (a-b)/(a+b)$, in which, a is half long axis, and b is short one.

The elastic displacement field can be solved from following equation[12]:

$$2G(u+iv) = k\varphi(\zeta) - \frac{\omega(\zeta)}{\omega'(\zeta)}\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)} \tag{6}$$

where $k = \begin{cases} 3-4v & \text{plane stress} \\ (3-v)/(1-v) & \text{plane strain} \end{cases}$

On the surface of crack $\rho = 1$, $\zeta = e^{i\theta} = S$, the resultant between arbitrary two points on S is expressed as[7]

$$(X+iY)_{AB} = -i\left[\varphi(\zeta) + \omega(\zeta)\frac{\overline{\varphi'(\zeta)}}{\overline{\omega'(\zeta)}} + \overline{\psi(\zeta)}\right]_A^B \tag{7}$$

Because surface of the crack is free, that is

$$(X+iY)_S = 0$$

we have

$$-\overline{\psi(\zeta)} = \varphi(\zeta) + \omega(\zeta)\frac{\overline{\varphi'(\zeta)}}{\overline{\omega'(\zeta)}} \tag{8}$$

Substituting Eqn. (8) into Eqn. (6), a new elastic displacement equation of an elliptical crack is obtained as

$$2G(u+iv) = (k+1)\varphi(\zeta) \tag{9}$$

Considering pure compression and pure shear states respectively, and using superimposition principle, the complex stress function under compressive-shear loading can be taken as

$$\varphi^{(1)}(S) = -\frac{R}{4}(\sigma_y - \sigma_n)[e^{i\theta} - (-2+m)e^{-i\theta}] \tag{10}$$

$$\varphi^{(2)}(S) = iR(\tau_{xy} + \tau_f)e^{-i\theta} \tag{11}$$

Substituting Eqns. (10) and (11) into (19), and using the trigonometric relation $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$, we get

$$u = -\frac{R}{E}(\sigma_y - \sigma_n)(3-m)\cos\theta + \frac{4R}{E}(\tau_{xy} + \tau_f)\sin\theta \tag{12}$$

$$v = -\frac{R}{E}(\sigma_y - \sigma_n)(1-m)\sin\theta + \frac{4R}{E}(\tau_{xy} + \tau_f)\cos\theta \tag{13}$$

This is the surface displacement solution of an elliptical crack under compressive-shear loading. specific surface energy of the solid-liquid interface, L is the latent heat of solidification.

3 CLOSING LAW AND CRITICAL COMPRESSIVE STRESS OF ELLIPTICAL CRACK

It is possible for an elliptical crack to be in unclosing state, or in full closing state, and or in local closing state. In geometry, the corresponding displacement condition are as follows:

1) closing at ends

$$v = 0, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = 0, \quad \theta = 0, \pi \tag{14}$$

2) closing at middle

$$v = -R(1-m)\sin\theta, \quad \theta \neq 0, \pi \tag{15}$$

Here sine curve is used to simulate the surface of an elliptical crack.

3) closing in whole

$$v = -R(1-m)\sin\theta, \quad 0 \leqslant \theta \leqslant \pi \tag{16}$$

If let

$$A = -\frac{(\sigma_y - \sigma_n)}{E}R(1-m) \tag{17}$$

$$B = \frac{4(\tau_{xy} + \tau_f)}{E}R \tag{18}$$

the critical stresses of three closing models are as following respectively:

1) closing at ends

Substituting Eqn. (13) into (14), we have

$$\begin{cases} A\sin\theta + B\cos\theta = 0 \\ A\cos\theta - B\sin\theta = 0 \end{cases} \tag{19}$$

When $\theta = 0$, π, solving Eqn. (19) we have $B = 0$, $A = 0$. Because the surfaces of crack do not touch each other, so $\sigma_n = 0$. Thus, the critical stress when crack closes at ends are

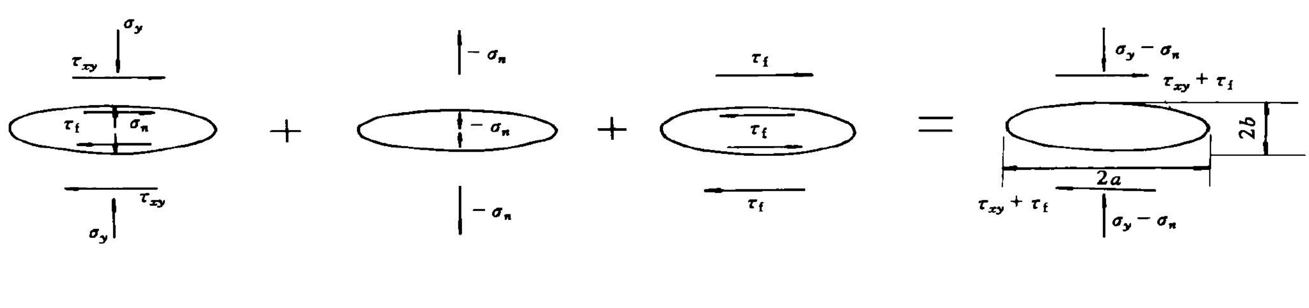

Fig. 2 Stress superimposition method for an elliptical crack

$$\sigma_y = 0,\quad \tau_{xy} = 0 \tag{20}$$

From Eqns. (3) and (4), we know that Eqn. (20) expresses the condition of zero compressive loading.

2) closing at middle

Substituting Eqn. (13) into Eqn. (15), we have

$$A\sin\theta + B\cos\theta = -R(1-m)\sin\theta,\quad \theta \neq 0,\ \pi \tag{21}$$

or

$$(A+b)\sin\theta + B\cos\theta = 0,\quad \theta \neq 0,\ \pi \tag{22}$$

Solving Eqn. (22), we get

$$\theta_c = \tan^{-1}\left\{\frac{4(\tau_{xy}+\tau_f)}{(1-m)[(\sigma_y-\sigma_n)-E]}\right\} \tag{23}$$

This is the initial closing place of an elliptical crack under compressive-shear loading. At this moment, it is still believed that $\sigma_n = 0$. Substituting $\sigma_y = \sigma_c$ into Eqn. (23), we get the critical compressive stress when crack closes at middle:

$$\sigma_c = E + \frac{4\tau_{xy}}{(1-m)\tan\theta_c} \tag{24}$$

It is seen from above equation that the critical compressive stress is relative to shear stress.

3) closing in whole

Substituting Eqn. (13) into Eqn. (16), we have

$$A\sin\theta + B\cos\theta = -R(1-m)\sin\theta,\quad \text{all}\ \theta \tag{25}$$

Solution of Eqn. (25) is $A = -R(1-m)$, and $B = 0$, from these, we can get

$$\sigma_y - \sigma_n = E \tag{26}$$

$$\tau_{xy} = -\tau_f = -\mu\sigma_n \tag{27}$$

Using Eqn. (1), the critical compressive stress when crack closes in whole is as follows:

$$\sigma_c^* = E \tag{28}$$

Eqn. (28) shows that the exerted compressive stress must reach elastic modulus of materials in order to close the crack in whole. In other words, it is impossible for an oblique elliptical crack under compressive loading to close at ends. So there must exist the local closing state of crack.

From above analysis, following results can be obtained.

1) The closure of an elliptical crack under compressive-shear stress is a process, in which the closing face gradually extend. So the critical compressive stress is not single.

2) It is impossible for an elliptical crack to close in whole, the closing is just local, so, there must exists the compressive stress intensity factor, and it must affects the extension of crack.

3) Shear stress is very important in the closing process of crack. It determines the initial closing place of crack, and governs the magnitude of critical compressive stress. That is, the greater the shear stress, the smaller the critical load, and it is the same vice versa.

4) When subjected to pure compressive loading ($\tau_{xy} = 0$), the crack is either open or close, no transition period exists, the critical load is just one, which can be proved as follows.

Because $\tau_{xy} = 0$, so we have $B = 0$. Thus, Eqn. (21) becomes

$$A\sin\theta = -R(1-m)\sin\theta \tag{29}$$

Solution of above equation is

$$\sin\theta = 0,\quad \theta = 0,\ \pi \tag{30}$$

$$A = -R(1-m),\quad \sigma_c^* = E \tag{31}$$

However, the strength of most brittle materials is much less than its elastic modulus. So, before crack closes in whole, the material has already broken down.

5) When subjected to pure shear loading ($\sigma_y = 0$), it is possible for a crack to close at one point, which is proved as follows.

Because $\sigma_y = 0$, and $\sigma_n = 0$, so we have $A = 0$. Thus, Eqn. (21) becomes

$$B\cos\theta = -R(1-m)\sin\theta \tag{32}$$

Solution of above equation is

$$\sin\theta = 0 \text{ and } B = 0,\ \text{natural state} \tag{33}$$

or

$$\theta_0 = \tan^{-1}\left[\frac{4\tau_{xy}}{(1-m)E}\right],\ \text{local closure} \tag{34}$$

4 EFFECT OF CLOSING CRACK ON STRESS INTENSITY FACTOR

Once crack closes (either in whole or localization), it can be regarded as the mathematical one ($m = 1$). So, the stress intensity factor at the tip of the crack can be written as follows:

$$K = K_{\rm I} + iK_{\rm II} = 2\sqrt{2}\lim_{\zeta\to\zeta_1}\{\omega(\zeta)-\omega(\zeta_1)\}^{\frac{1}{2}}\varphi'(\zeta)/\omega'(\zeta) \tag{35}$$

Substituting Eqns. (10) and (11) into Eqn. (35), and dividing the real part and imaginary part, we get

$$K_{\rm I} = -(\sigma_y - \sigma_n)\sqrt{\pi a} \tag{36}$$

$$K_{\rm II} = (\tau_{xy} - \tau_f)\sqrt{\pi a} \tag{37}$$

1) When crack opens, $\sigma_n = 0$

$$K_{\rm I} = -\sigma_y\sqrt{\pi a} \tag{38}$$

$$K_{\rm II} = \tau_{xy}\sqrt{\pi a} \tag{39}$$

It is seen that the compressive stress intensity factor is equal to the negative value of tensive stress intensity factor only when crack is open.

2) When crack closes in localization, substituting Eqn. (24) into Eqns. (36) and (37), we get

$$K_{\rm I} = -\left[E + \frac{4\tau_{xy}}{(1-m)\,{\rm tg}\,\theta_c}\right]\sqrt{\pi a} \tag{40}$$

$$K_{\rm II} = \left\{\tau_{xy} + \mu\left[\sigma_y + \left(E + \frac{4\tau_{xy}}{(1-m)\,{\rm tg}\,\theta_c}\right)\right]\right\}\sqrt{\pi a} \tag{41}$$

Above equations show that when crack is close in localization, the compressive stress intensity factor is determined by τ_{xy}, not by σ_y. So, it is recognized that the compressive stress intensity factor is never equal to the negative value of tensive stress intensity factor when crack closes. It is a new mechanical quantity distinguishing from the traditional stress intensity factors of Ⅰ, Ⅱ and Ⅲ mode.

It needs to be pointed out that the results given here are obtained without considering the condition of long axial displacement of an elliptical crack. In spite of this, they still reflect the law of closing crack and stress intensity factor to some extent.

5 CONCLUSIONS

1) Either closing in localization ($\tau_{xy} \neq 0$) or in whole ($\tau_{xy} = 0$) can occur in the elliptical crack under compressive-shear loading, in which shear stress is a dominant factor.

2) The critical compressive stress is determined by shear stress when crack closes in localization. While the critical compressive stress is a constant under pure compressive loading, and its value is equal to elastic modulus of brittle material.

3) There always exists the compressive stress intensity factor at tip of crack when it closes in localization, and its value is determined by shear stress, not by compressive stress.

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