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Elastic-plastic analytical solution for centric crack loaded by two pairs of point shear forces in finite plate

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Abstract: The near crack line analysis method was used to investigate a centric crack loaded by two pairs of point shear forces in a finite plate, and the analytical solution was obtained. The solution includes the unit normal vector of the elastic-plastic boundary near the crack line, the elastic-plastic stress fields near the crack line, the variations of the length of the plastic zone along the crack line with an external load, and the bearing capacity of a finite plate with a centric crack loaded by two pairs of point shear forces. The results are sufficiently precise near the crack line because the assumptions of small scale yielding theory have not been made and no other assumptions are taken.

Key words: point shear forces; finite plate; centric crack; near crack line analysis method; elastic-plastic analytical solution

1 Introduction

A great number of cracks exist in metal materials. Their existence and interaction often lead to high stress concentration and become the source of weakening and failure of metal materials[1−4]. The elastic-plastic analysis for a cracked plate with finite dimensions is one of the most difficult fields of elastic-plastic mechanics. To analyze and describe failure of an elastoplastic material containing a centric crack loaded by two pairs of shear point forces in a finite body, various different methods have been developed. Among others, near crack line analysis method has proved its usefulness in many applications. Near crack line analysis method was originally proposed by ACHENBACH and LI[5] and was further studied by ACHENBACH and DUNAYEVSKY[6], and GUO and LI[7]. However, the earlier crack line field analysis method solves the problem by matching a specific solution of plastic field near crack line with elastic singular *K* field near the crack line, for which the conventional small scale yielding condition must be taken. Based on works by ACHENBACH and LI[5], and GUO and LI[7], the crack line analysis method with novel significance has been proposed, which fundamentally breaks through the limitation of the traditional small scale yielding conditions[8−11]. The near crack line elastic-plastic stress fields for a centric crack loaded by two pairs of tensile and shear point forces in an infinite body of elastic-plastic material was researched by using the improved near crack line analysis method[12,13]. The coalescence mechanism of splitting failure of crack-weakened rock subjected to compressive loads was analyzed by using the improved near crack line analysis method[14]. The near crack line elastic-plastic stress fields for an eccentric crack loaded by a pair of tensile point forces in an elastic-plastic material was studied by using the improved near crack line analysis method[15].

In this paper, the improved near crack line is extended to the problem of mode II crack loaded by two pairs of point shear forces in a finite body in an elastic-plastic material . This problem often exists in the practical engineering and can be easily investigated by experiment, so the investigation of problem is important both theoretically and in applications. The research in this paper may be helpful to provide an insight into the failure behavior of elastic-plastic material containing a centric crack loaded by two pairs of point shear forces in a finite plate.

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2 Theoretical model

2.1 Basic equations

Consider a crack loaded by two pairs of point shear force in an elastic-perfectly plastic material as shown in Fig.1. x_1 and x_2 are stationary coordinate system with x_1 axis parallel to the crack front. A moving coordinate system x, y is centered at the crack tip with its axis parallel to the x_1 and x_2 axes. For a state of plane stress, the σ_z, τ_{xz} and τ_{yz} vanish identically, hence the equilibrium equations are

$$\begin{cases}\dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{xy}}{\partial y}=0\\ \dfrac{\partial\tau_{xy}}{\partial x}+\dfrac{\partial\sigma_y}{\partial y}=0\end{cases} \tag{1}$$

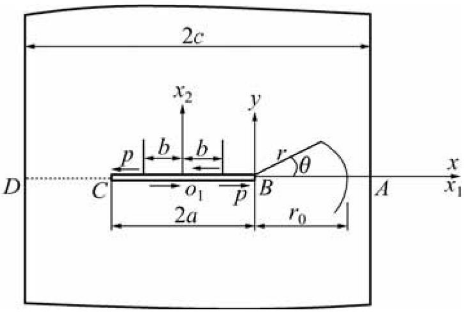

Fig.1 Schematic diagram of centric crack loaded by two pairs of point shear forces in finite plate

The Huber-Mises yield criterion is

$$\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\tau_{xy}^2=3k^2 \tag{2}$$

where k is the yield stress in pure shear .

In the plastic loading zone near the crack line x/y 1, $0< x \leqslant x_p$, where $x=x_p$ defines the elastic-plastic boundary on the crack line. The stress can be expressed by[9]

$$\begin{cases}\sigma_x=f_1(x)y+O(y^3)\\ \sigma_y=m_1(x)y+O(y^3)\\ \tau_{xy}=s_0(x)+s_2(x)y^2+O(y^4)\end{cases} \tag{3}$$

Here we have taken into account that σ_x and σ_y are antisymmetric with respect to y=0, while τ_{xy} is symmetric with respect to y=0. Substituting the expansion (3) into equilibrium Eqn.(1), the Huber Mises yield criterion (2), yields a system of ordinary differential and collecting terms of the same order y equations for the coefficients of power series expansions of stress components, and solving the equation system gives

$$\begin{cases}\sigma_x=-3k\dfrac{y}{x+L}+O(y^3)\\ \sigma_y=0\\ \tau_{xy}=k-1.5k\dfrac{y^2}{(x+L)^2}+O(y^4)\end{cases} \tag{4}$$

where L is integral constant. Eqn.(4) is the general solution of stresses of the plastic region near the crack line.

2.2 Elastic stress field and elastic-plastic boundary near crack line

Near the crack line the stress components in the plastic zone should be matched with the precise elastic stress field. For a mode II crack in an infinite plate corresponding to the finite plate in Fig.1, the Westergaard's complex stress function is

$$Z_{\mathrm{II}}(z)=\frac{2pz\sqrt{a^2-b^2}}{\pi(z^2-b^2)\sqrt{z^2-a^2}} \tag{5}$$

The stress components are written as

$$\begin{cases}\sigma_x=2\,\mathrm{Im}\,Z_{\mathrm{II}}(z)+y\,\mathrm{Re}\,Z'_{\mathrm{II}}(z)\\ \sigma_y=-y\,\mathrm{Re}\,Z'_{\mathrm{II}}(z)\\ \tau_{xy}=\mathrm{Re}\,Z_{\mathrm{II}}(z)-y\,\mathrm{Im}\,Z'_{\mathrm{II}}(z)\end{cases} \tag{6}$$

where

$$Z'_{\mathrm{II}}(z)=\frac{\mathrm{d}Z_{\mathrm{II}}(z)}{\mathrm{d}z}$$

$$z=x_1+\mathrm{i}x_2(\mathrm{i}=\sqrt{-1})$$

By transforming the Cartesian coordinate system to polar coordinate system centered at the crack tip, the precise elastic stresses can be expanded in a power series of the angle to the crack line as

$$\sigma_x=-6AR\theta+O(\theta^3) \tag{7}$$

$$\sigma_y=2AR\theta+O(\theta^3) \tag{8}$$

$$\tau_{xy}=\frac{2p(a+r)\sqrt{a^2-b^2}}{\pi[r(2a+r)]^{1/2}[(a+r)^2-b^2]}-$$

$$\frac{p}{B}\sqrt{a^2-b^2}\{5a^7+40ra^6+8r^7+$$

$$a^5(138r^2-10b^2-\frac{44b^2r}{a})+a^3(258ar^3+5b^4-54b^2r^2)+a^2(279ar^4+4b^4r-22b^2r^3)+174a^2r^5-2ab^2r^4+58ar^6\}\theta^2+O(\theta^4) \tag{9}$$

where

$$A=\frac{pr\sqrt{a^2-b^2}}{\pi[r(2a+r)]^{3/2}[(a+r)^2-b^2)]^2}$$

$$R=a^4+6a^3r+8ar^3+2r^4-a^2(b^2-11r^2)$$

$$B=\pi[r(2a+r)^5]^{1/2}[(a+r)^2-b^2]^3$$

For a finite plate with a centric crack loaded by two pairs of point forces $2p$ as shown in Fig.1. The elastic stresses satisfying all the boundary conditions have not been obtained. But we may consider that its stresses near the crack line are similar to the forms of Eqns.(7)−(9) of the corresponding infinite plate. Then the stresses near crack line for the finite plate that is shown in Fig.1 may be modified from Eqns.(7)− (9) as

$$\sigma_x=-\frac{6QAR\theta}{p}+O(\theta^3) \tag{10}$$

$$\sigma_y=\frac{2QAR\theta}{p}+O(\theta^3) \tag{11}$$

$$\tau_{xy}=\frac{2Q(a+r)\sqrt{a^2-b^2}}{\pi[r(2a+r)]^{1/2}[(a+r)^2-b^2]}-\frac{Q\sqrt{a^2-b^2}}{B}\cdot\{5a^7+40a^6r+8r^7+a^4(138ar^2-10ab^2-44b^2r)+a^3(258ar^3+5b^4-54b^2r^2)+a^2(279ar^4+4b^4r-22b^2r^3)+a(174ar^5-2b^2r^4+58r^6)\}\theta^2+O(\theta^4) \tag{12}$$

where

$$A=\frac{pr\sqrt{a^2-b^2}}{\pi[r(2a+r)]^{3/2}[(a+r)^2-b^2)]^2}$$

$$R=a^4+6a^3r+8ar^3+2r^4-a^2(b^2-11r^2)$$

$$B=\pi[r(2a+r)^5]^{1/2}[(a+r)^2-b^2]^3$$

where Q is a parameter related to the dimension $2c$, the length of the crack $2a$ and the point forces $2p$, etc.

Obviously, Eqns.(10)−(12) satisfy the condition that the crack surface is traction free, because they are modified by a parameter Q from Eqns.(7)−(9) are expanded from the exact Eqn.(6) which satisfy the boundary condition that the crack surface is traction free. Now if a reasonable boundary condition on the crack line is introduced, the Eqns.(10)−(12) are valid near the crack line and parameter Q can be determined.

The boundary condition on the crack line can be established by cutting the plate in two along the crack lines as shown in Fig.1. We just consider the equilibrium of a part. In order that the stresses on the crack line are in equilibrium with the loading $2p$ we have

$$2kr_0+2\int_{r_0}^{c-a}(\tau_{xy})_{\theta=0}\mathrm{d}r=2p \tag{13}$$

Substituting Eqns.(10)−(12) into Eqn.(13) yields

$$Q=\frac{\pi}{t}(p-kr_0) \tag{14}$$

where

$$t=\ln\frac{G\left(\sqrt{c^2-b^2}-\sqrt{a^2-b^2}\right)}{\left(\sqrt{c^2-b^2}+\sqrt{a^2-b^2}\right)},$$

$$G=\frac{\sqrt{(a+r_0)^2-b^2}+\sqrt{a^2-b^2}}{\sqrt{(a+r_0)^2-b^2}-\sqrt{a^2-b^2}}$$

2.3 Matching results near crack line at elastic-plastic boundary

The elastic-plastic boundary is defined by $r=r_p(\theta)$, since $r_p(\theta)$ is symmertric with respect to $\theta=0$, for small θ we have (Fig.2)

$$r_p(\theta)=r_0+r_2\theta^2 \tag{15}$$

It follows from Eqn.(15) that the unit normal vector $n=(n_x,n_y)$ of the elastic-plastic boundary is

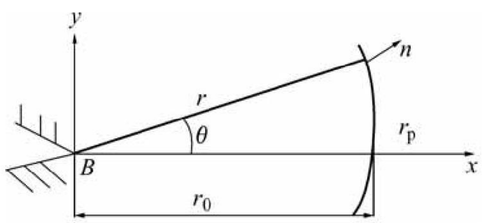

Fig.2 Schematic diagram of elastic-plastic boundary

$$\begin{cases} n_x=1-\frac{1}{2}B_1^2\theta^2 \\ n_y=B_1\theta \end{cases} \tag{16}$$

where

$$B_1 = 1 - \frac{2r_2}{r_0}$$

At the elastic-plastic boundary, by use of Eqn.(15), we have

$$\begin{cases} x = r_0 + r_0\left(\dfrac{r_2}{r_0} - \dfrac{1}{2}\right)\theta^2 \\ y = r_0\theta \end{cases} \tag{17}$$

Substituting Eqn.(15) into Eqns.(10)−(12), the stress field on the side of the elastic zone can be obtained as

$$\sigma_x = -\frac{6QA_0R_0\theta}{p} + O(\theta^3) \tag{18}$$

$$\sigma_y = \frac{2QA_0R_0\theta}{p} + O(\theta^3) \tag{19}$$

$$\tau_{xy} = \frac{2Q(a+r_0)\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{1/2}[(a+r_0)^2-b^2]} - \frac{Q}{B_0}\sqrt{a^2-b^2}\cdot$$
$$\{5a^7+40a^6r_0+8r^7+a^5(138r_0^2-10b^2-\frac{44b^2r_0}{a})+$$
$$a^3(258ar_0^3+5b^4-54b^2r_0^2)+a^2(279ar_0^4+4b^4r_0-$$
$$22b^2r_0^3)+a(174ar_0^5-2b^2r_0^4+58r_0^6)\}\theta^2+$$
$$\frac{2QA_0R_0\theta^2}{p}+O(\theta^4) \tag{12}$$

where

$$A_o = \frac{pr_0\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{3/2}[(a+r_0)^2-b^2)]^2}$$

$$R_0 = a^4+6a^3r_0+8ar_0^3+2r_0^4-a^2(b^2-11r_0^2)$$

$$B_0 = \pi[r_0(2a+r_0)^5]^{1/2}[(a+r_0)^2-b^2]^3$$

At the elastic-plastic boundary, the expressions of σ_{nn} and σ_{ns} are

$$\begin{cases} \sigma_{nn} = \sigma_x n_x^2 + \sigma_y n_y^2 + 2\tau_{xy}n_x n_y \\ \sigma_{ns} = (n_x^2 - n_y^2)\tau_{xy} + (\sigma_y - \sigma_x)n_x n_y \end{cases} \tag{21}$$

Substituting Eqns.(4), (16) and (17) into (21), the traction components σ_{nn} and σ_{ns} on the side of the plastic zone can be obtained. Substituting Eqns.(16), (18), (19) and (20) into (21) , the traction components σ_{nn} and σ_{ns} on the side of the elastic zone can be obtained. From the condition that σ_{nn} and σ_{ns} are continuous across the elastic-plastic boundary, and by collecting terms of the same order of θ^o, θ^1 and θ^2 we can obtain r_0, L, B_1:

$$k = \frac{2(p-kr_0)(a+r_0)\sqrt{a^2-b^2}}{t[r_0(2a+r_0)]^{1/2}[(a+r_0)^2-b^2]} \tag{22}$$

$$-\frac{3kB_1r_0}{r_0+L} - \frac{1.5kr_0^2}{(r_0+L)^2} - 2B_1^2k =$$
$$\frac{2Q(-2B_1^2+1)(a+r_0)\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{1/2}[(a+r_0)^2-b^2]} - \frac{Q}{B_0}\sqrt{a^2-b^2}\cdot$$
$$\{5a^7+40r_0a^6+8r_0^7+a^5(138r_0^2-10b^2-\frac{44b^2r_0}{a})+$$
$$a^3(258ar_0^3+5b^4-54b^2r_0^2)+a^2(279ar_0^4+4b^4r_0-$$
$$22b^2r_0^3)+a(174ar_0^5-2b^2r_0^4+58r_0^6)\}-$$
$$\frac{Q(7B_1r_0+1)A_0R_0}{p} \tag{23}$$

$$2B_1k - \frac{3kr_0}{r_0+L} = -\frac{6QA_0R_0}{p} + \frac{4B_1Q(a+r_0)\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{1/2}[(a+r_0)^2-b^2]} \tag{24}$$

where

$$A_0 = \frac{pr_0\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{3/2}[(a+r_0)^2-b^2)]^2}$$

$$R_0 = a^4+6a^3r_0+8ar_0^3+2r_0^4-a^2(b^2-11r_0^2)$$

From Eqn.(22), we can find the length of the plastic region on the crack line. Substituting Eqn.(22) into Eqns.(23) and (24) we can obtain the analytical solutions L and B_1 .

By introducing three dimensionless variable $\xi=r_0/a$, $\eta=2p/(2ak)$, $\lambda=b/a$, $m=c/a$, Eqn.(22) can be rewritten by

$$\eta = \xi + \frac{t(2\xi+\xi^2)^{1/2}\rho^2}{2(1+\xi)\sqrt{1-\lambda^2}} \tag{25}$$

where

$$t = \ln\frac{(\sqrt{m^2-\lambda^2}-\sqrt{1-\lambda^2})(\rho+\sqrt{1-\lambda^2})}{(\sqrt{m^2-\lambda^2}+\sqrt{1-\lambda^2})(\rho-\sqrt{1-\lambda^2})}$$

$$\rho^2 = (1+\xi)^2 - \lambda^2$$

According to Eqn.(22), the bearing capacity of a finite plate with a centric crack loaded by two pairs of point shear forces can be determined. The numerical results are shown in Fig.3.

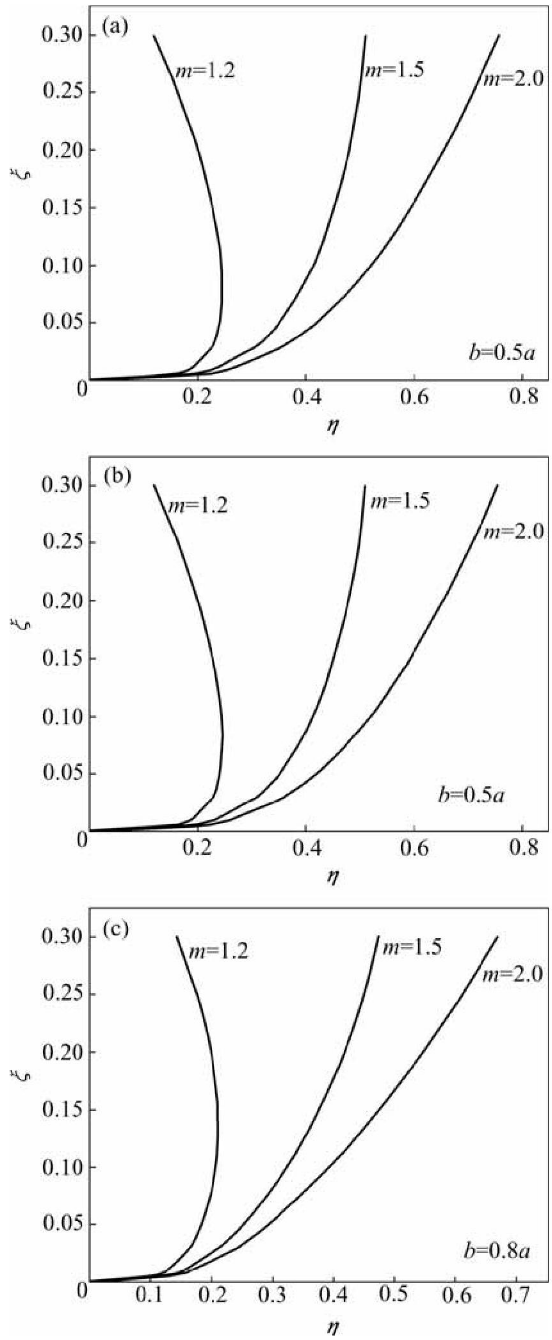

Fig.3 Variations of r_0/a (η) with $2p/2ak$ (ξ) under different b and m

It is shown in Fig.3 that the bearing capacity of finite plate with a centric crack loaded by two pairs of point shear forces is sensitive to different values of m and λ. The predicted bearing capacity is higher as value of m is larger. It is seen for m=1.2 that the bearing capacity of a finite plate will decrease when the external loads reach critical value. The above result implies that the crack will experience unstable growth when the external loads reach critical value.

3 Results and discussion

From Eqns.(22) or (25), the lengths of the plastic zone on the crack line, r_0 or ξ can be determined.

L, the integral constant in expressions of the plastic stresses, can be found from Eqns.(23) and (24). In the stress fields of plastic zone there are not usually singularities near crack tip because of the integral constant L. However, if $L\leqslant 0$, stress in plastic zone near the crack tip may display singularities. The singularity is not different from the one in the classical case or small scale yielding case because it appears at any position on the crack line, but also the crack might not keep in static which might fall into crack initiation and propagation or buckling propagation. On the other hand, if the integral constant $L<0$, unacceptably, the strains near the crack tip ($x\rightarrow 0$) will be negative. Therefore, in this paper, we must take $L>0$, and L=0 is a limit case. So according to Eqns.(23) and (24) we can find out the maximum lengths of the plastic zone which are related to the position of point forces. The corresponding maximum point forces at different position of the point forces on the crack surface can be determined by Eqn.(22).

By Eqns.(23) and (24), the unit normal vector of the elastic-plastic boundary near the crack line is obtained, so the probable change of the whole plastic zone can be predicted.

When $(c-a)/a$ or $m-1$ less than the value ξ_{max} in Table 1, the dimensionless limit length of plastic zone on the crack line is $m-1$.

Table 1 Maximum loads $2p_{max}/2ak$ corresponding to different λ values

λ	ξ_{max}	c/a=2	b/a=1	$b/a=\infty$
0	0.618	0.973	1.897	2.104
0.1	0.613	0.971	1.883	2.087
0.2	0.596	0.963	1.842	2.038
0.3	0.567	0.948	1.77	1.953
0.4	0.527	0.923	1.665	1.830
0.5	0.472	0.833	0.524	1.665
0.6	0.404	0.821	0.338	1.452
0.7	0.319	0.724	1.099	1.181
0.8	0.218	0.573	0.792	0.846
0.9	0.108	0.342	0.427	0.445

4 Conclusions

Crack line field analysis method greatly simplifies the complexity of crack elastic-plastic problem and overcomes the difficulty in mathematics, which transforms the partial differential equation into ordinary differential equation. As have been done above, the near crack line elastic stress, satisfies the far field boundary conditions and the boundary conditions of the crack surfaces, has been used to match with the general solutions of the plastic region near the crack line, and no assumptions have been made during the analyses, so the analyses are precise and not confined by small scale yielding conditions. The stress fields of plastic zone are not usually singularities near crack tip because of the integral constant L.

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